Mark–to–Market Reinsurance and Portfolio Selection: Implications for Information Quality*

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Abstract

This paper investigates the optimal mark–to–market reinsurance and asset investment strategies for insurers with complete or partial information on expected return. The insurer with partial information is assumed to have prior belief on the expected return and to update her posterior beliefs by exploiting its price information. We show that the strategies of the insurer with partial information can be highly dependent on prior belief, and that variation in posterior beliefs gives rise to her counter–cyclical investment demand. By comparing the two insurers' strategies, we show that insurer's utility gain by the information acquisition is a concave function with respect to prior belief. This conclusion can be explained by the relative importance between reinsurance costs and demands on precautionary saving.

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1. Introduction

Insurance companies and pension funds have paid much attention to asset–liability management, during the recent financial crisis in particular. For example, the OECD Insurance and Private Pension Committee has advised insurers to not only meet liability obligations to policyholders but also to mitigate several risk exposures. This new regulatory trend gives rise to risk–based capital requirements along with quantitative and qualitative provisions (OECD, 2015). A central aspect of the capital requirements is to determine proper *discount rates* (or expected returns) used to adjust assets and liabilities on a market basis, namely a *mark–to–market valuation*.

Discount rates vary over time (Cochrane, 2011). The capital requirements subject to variation in expected returns can cause portfolio manages to become increasingly conservative. For example, Solvency II in European countries established a guideline that capital must cover unexpected losses over a one–year horizon with a probability 99.5%. Even with strong savings accumulations, insurers, traditionally recognized as long–term institutional investors, also desire to increase their reserve funds against future potential losses. Indeed, investment strategy can be used together with reinsurance strategy as an important hedging tool in reducing insurance claims risk. Motivated by the recent regulatory trend, this paper studies the optimal mark–to–market reinsurance and asset allocation strategies for risk-averse insurers.

Two different types of models are considered: a model for an insurer with complete information on the expected return from risky investment (the CI model) and a model for an insurer with partial information (the PI model). Basically, we assume that the expected return evolves by following a two–state hidden Markov chain, but that return volatility does not change over time.[1] The insurer under the PI model is assumed not to exactly observe time variations in the expected return, but to update her beliefs on the expected return by exploiting information on the risky asset prices. The insurers are assumed to have utility preference of constant absolute risk aversion (CARA) type and aim to maximize their utility by controlling the mark–to–market proportional reinsurance rates and risky investment amount.

Precisely, this paper is a rare research finding an intersection between studies on *information quality* in asset pricing theory and those on insurer's optimal behavior.[2] Conventional studies

[1] Merton (1980) shows that expected return is harder to estimate than is return volatility.

[2] A large body of the literature has provided evidence for stochastically changing investment opportunities driven by the first and second moments of risky assets or stochastic real interest rates (Korn and Kraft, 2002; Campbell et al., 2004; Chacko and Viceira, 2005). In this regard, information–quality implications have been extensively studied in asset pricing theory (David, 1997; Ai, 2010) and on optimal portfolio choice (Honda, 2003; Liu, 2011).

have commonly assumed that insurers have complete information on insurance/reinsurance and financial markets,[3] which incurs no estimation risk over the indefinite future. Accordingly, the insurers considered in the existing literature can only make myopic decisions without a hedging motive under *complete information.* To see this clearly, we introduce the CI model which employs a two–state regime-switching asset price process, and compare the results by the CI model with those by the PI model.

This paper has three main contributions to the literature. First, we find an explicit representation of *certainty equivalent wealth* (CEW) for the insurer with partial information. We show that the CEW can be decomposed into two components: a components by reinsurance costs and the other component by the insurer's demands on precautionary saving. This representation can help better understand her mark–to–market reinsurance and investment strategies in a logical way. For instance, the decomposition leads us to conclude that

- an insurer with low risk aversion can be more affected by the demand on precautionary saving than reinsurance costs, and
- a large correlation between financial assets and insurance claims improves a risk–sharing effect.

Second, we show that the mark–to–market reinsurance strategy under the PI model depends on insurer's prior belief on a pro–cyclical basis. However, we show that the mark–to–market investment strategy can be counter–cyclical. The counter–cyclical investment strategy, caused by changes in future investment opportunities, can help enhance the risk–based capital requirements by reducing potential pro–cyclical overreaction, especially during economic recessions (OECD, 2015).

Third, by comparing the two insurers' strategies, we show that insurer's utility gain by the information acquisition is a concave function with respect to prior belief. This conclusion can be explained by the relative importance between reinsurance costs and demands on precautionary saving.

The rest of the paper is organized as follows. Section 2 provides our basic setting and formulates the insurer's problems. Sections 3 and 4 introduce the CI model and the PI model and show insurer's optimal strategies, respectively. Section 5 provides numerical implication in relation to information quality by comparing the two models. Section 6 concludes.

[3] See Cao and Wan (2009), Gu et al. (2010), and Liang et al. (2011).

2. The Basic Set-Up

2.1. Financial Market

The financial market is frictionless and has two investment assets, one risk–free asset (bond) and one risky asset (stock). The risk–free asset price, B_t, evolves by

$$dB(t) = rB(t)dt,$$

with a constant instantaneous interest rate r. The return of the risky asset evolves by

$$dR(t) = \mu(t)dt + \sigma dW_R(t), \tag{1}$$

where $\mu(t)$ is a stochastic process, $\sigma > 0$ is a constant parameter, and $W_R(t)$ is a standard Brownian motion. An insurer considered here is a price-taker in the financial market.

2.2. Insurance Claim

The insurer pays insurance claims to policyholders as a return service of receiving some insurance premium. We assume there are sufficiently many policyholders so that the arrival of insurance claims are quite frequent. Thus, we can take the diffusion model in Iglehart (1969) as the cumulative insurance claims $C(t)$ satisfying

$$dC(t) = \alpha dt - \beta dW_C(t),$$

where $\alpha > 0$ is its expected growth rate, $\beta > 0$ is its constant volatility, and $W_C(t)$ is a standard Brownian motion. We assume the exogenous shocks on the stock returns and the insurance claims have a correlation of ρ, that is,

$$dW_R(t) \cdot dW_C(t) = \rho dt.$$

2.3. Reinsurance

A reinsurer can take all reinsurance demands of the insurer, and the insurer has a proportional reinsurance strategy. Given a time-t reinsurance rate as $\varepsilon(t)$, the reinsurer pays $\varepsilon(t)dC(t)$-dollar to policyholders for the infinitesimal time period of $[t, t + \Delta t)$ and the insurer pays the rest, i.e., $(1 - \varepsilon(t))dC(t)$.

We assume the insurer's safety loading is $\theta \geq 0$, and the reinsurer's safety loading is $\eta \geq 0$. Then, the insurance premium $p_I(t)$ must be

$$p_I(t) = (1+\theta)\alpha,$$

and the reinsurance premium $p_O(t)$ must be

$$p_O(t) = (1+\eta)\alpha\varepsilon(t).$$

We take the assumption of $\eta \geq \theta$; otherwise, the insurer's cheapest strategy would be to take a perfect reinsurance strategy, i.e., $\varepsilon(t) = 1$.

2.4. The Insurer's Goal

Taking all considerations together, the insurer's surplus process $S(t)$ satisfies

$$\begin{aligned} dS(t) &= p_I(t)dt - (1-\varepsilon(t))dC(t) - p_O(t)dt \\ &= (\theta - \eta\varepsilon(t))\alpha dt + \beta(1-\varepsilon(t))dW_C(t), \end{aligned}$$

and thus, the insurer's wealth, $X(t)$, evolves as

$$dX(t) = [rX(t) + (\mu(t)-r)\pi(t) + (\theta - \eta\varepsilon(t))\alpha]dt + \beta(1-\varepsilon(t))dW_C(t) + \sigma\pi(t)dW_R(t), \tag{2}$$

where $\pi(t)$ is the dollar investment amount in the risky asset.

The insurer's goal is to maximize her utility $U(\cdot)$ with respect to terminal wealth $X(T)$ by controlling reinsurance and portfolio strategies, $(\varepsilon(t),\, \pi(t))$. The insurer's utility preference is a CARA type:

$$U(x) = -\frac{1}{\gamma}e^{-\gamma x},$$

where γ is the coefficient of absolute risk aversion. In the long run, the goal is to find

$$\sup_{\varepsilon,\pi} E^{t,x}[U(X(T))],$$

under the condition of Equation (2). Here, $E^{t,x}[\cdot]$ is an expectation operator conditioned on the time-t wealth $X(t) = x$ under a real probability measure $\mathbb{P}$.

3. The CI Model: A Model with Complete Information

3.1. A Completely Informed Insurer

As a benchmark model, we first explore a model of optimal reinsurance and portfolio selection under complete information (henceforth, the CI model). We characterize the CI model as a two–state regime–switching model in which the expected return of the risky asset stochastically jumps between two states.

Suppose that $Y(t)$ follows a *observable* Markov chain with two states: *High regime* (or regime H), and *Low regime* (or regime L). In the CI Model, we take the assumption that the expected return of the risky asset satisfies

$$dR(t) = \mu_i dt + \sigma dW_R(t),$$

where $\mu_i \equiv \mu(Y(t) = i)$ for $i \in \{H, L\}$ satisfying $\mu_H > \mu_L$. Regime i jumps to regime j $(j \neq i)$ for $i, j \in \{H, L\}$ with intensity λ_i. Hence, for a infinitesimal length of time Δt, μ_i remains unchanged with probability of $1 - \lambda_i \Delta t$ or shifts to μ_j $(j \in \{H, L\},\ j \neq i)$ with probability of $\lambda_i \Delta t$. We assume that such Poisson-type jumps independently occur each other and are independent of any market risks $(W_R(t))$ and the insurance claims risk $(W_C(t))$.

In the CI model, all information concerning market risk, insurance claims risk, and regime risk is assumed to be immediately revealed to the insurer. Specifically, the insurer has full information confidence on the filtration of

$$\mathcal{F}(t) = \mathcal{F}^{W_R, Y}(t) \times \mathcal{F}^{W_C}(t),$$

where $\mathcal{F}^{W_R, Y}(t)$ stands for the filtration generated by information of the return shocks $W_R(u)$ and regime jumps $Y(u)$ up to time $u \leq t$ $(t \in [0, T])$, and $\mathcal{F}^{W_C}(t)$ is the filtration generated by the information of insurance claims shocks $W_C(t)$.

3.2. The Insurer's Optimal Strategies

The insurer's problem under the CI model is to find

$$V_i(t, x) = \sup_{\varepsilon_i, \pi_i} E^{t,x}[U(X(T)],$$

subject to

$$dX(t) = [rX(t) + (\mu_i - r)\pi_i + (\theta - \eta\varepsilon_i(t))\alpha]dt + \beta(1 - \varepsilon_i)dW_C(t) + \sigma\pi_i dW_R(t),$$

by taking the reinsurance and portfolio strategies (η_i, π_i) in each regime $i \in \{H, L\}$.

The dynamic programming principle yields a system of the two Hamilton–Jacobi–Bellman (HJB) equations for the value functions V_i:

$$\begin{aligned} V_{i,t} + \sup_{\varepsilon_i, \pi_i} \Big[&\{rx + (\mu_i - r)\pi_i + (\theta - \eta\varepsilon_i)\alpha\} V_{i,x} \\ &+ \frac{1}{2}\{\beta^2(1-\varepsilon_i)^2 + \sigma^2\pi_i^2 + 2\rho\beta(1-\varepsilon_i)\sigma\pi_i\} V_{i,xx} + \lambda_i(V_j - V_i) \Big] = 0 \end{aligned} \tag{3}$$

with the two terminal conditions

$$V_i(T, X(T)) = U(X(T)).$$

Here, $V_{i,t}$ and $V_{i,x}$ are respectively the first derivatives with respect to time t and wealth x, and $V_{i,xx}$ is the second derivative with respect to wealth. The first–order conditions lead us to the optimal strategies:

$$\varepsilon_i^* = \left[\frac{\alpha\sigma\eta - \rho\beta(\mu_i - r)}{\beta^2\sigma(1-\rho^2)}\right]\frac{V_{i,x}}{V_{i,xx}} + 1, \quad \text{and} \quad \pi_i^* = \left[\frac{\rho\alpha\sigma\eta - \beta(\mu_i - r)}{\beta\sigma^2(1-\rho^2)}\right]\frac{V_{i,x}}{V_{i,xx}}. \tag{4}$$

Theorem 1. *The value function in regime $i \in \{H, L\}$ is*

$$V_i(t, x) = -\frac{1}{\gamma} e^{-\gamma c(t)(x + f_i(t))},$$

where

$$c(t) = e^{r(T-t)},$$

and $f_i(t)$ satisfies

$$f_i'(t) - rf_i(t) + \frac{e^{-r(T-t)}}{\gamma}\lambda_i\left[1 - e^{-\gamma \exp\{r(T-t)(f_j(t) - f_i(t))\}}\right] + h_i(t) = 0, \quad f_i(T) = 0, \tag{5}$$

for $j \in \{H, L\}$ $(j \neq i)$ and

$$h_i(t) = \alpha(\theta - \eta) + \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\left[\frac{(\mu_i - r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\mu_i - r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\right].$$

Proof. See Appendix A. □

Theorem 1 shows that the values of $f_i(t)$ can examine the insurer's different behaviors across regimes.[4] In Section 5.4, we will compare $f_i(t)$ derived from the CI model with $f(t,p)$ derived from a partial information model (henceforth, the PI model).

4. The PI Model: A Model with Partial Information

4.1. A Partially Informed Insurer

Contrary to the CI model, the model with partial information (the PI model) takes the assumption that the insurer does not know exactly the current regime $Y(t)$ and, thus, the corresponding expected rate of risky return, $\mu(t)$. We assume that the insurer can infer the current regime from the market information.

In this case, the information set the insurer access can be represented as

$$\mathcal{G}(t) = \mathcal{F}^{W_R}(t) \times \mathcal{F}^{W_C}(t)\},$$

where $\mathcal{F}^{W_R}(t)$ is the filtration generated by the past history of the realized risky returns.[5] Instead, she exploits her own *prior* belief obtained from the market information, $\mathcal{F}^{W_R}(t)$, in order to infer the current regime of the financial market. We define the time–t prior belief, $p(t)$, as the probability of regime H inferred from the past information:

$$\text{probability of} \quad \{Y(t) = H|\, \mathcal{G}(t)\}.$$

Accordingly, the resulting filtered expected return $\bar{\mu}(t)$ corresponds to a weighted average of μ_i for $i \in \{H, L\}$:

$$\bar{\mu}(t) \equiv E\left[\mu(t)|\mathcal{G}(t)\right] = p(t) \cdot \mu_H + (1 - p(t)) \cdot \mu_L = \mu_L + (\mu_H - \mu_L)p(t).$$

We now can apply a non–linear filtering theory in Liptser and Shiryaev (2001) so that all the parameters are adapted to the filtration $\mathcal{G}(t)$. We employ an *innovation* process generated by the insurer's prior belief on the current regime as

$$\widehat{W}_R(t) = \int_0^t \frac{dR(s) - \bar{\mu}(s)ds}{\sigma}.^{6}$$

[4] We found f_i by utilizing a simple numerical scheme.

[5] Clearly, $\mathcal{F}^{W_R}(t) \subset \mathcal{F}^{W_R,Y}(t)$, implying that the information set of the CI model is bigger than that of the PI model.

[6] By the Girsanov theorem, $\widehat{W}_R(t)$ serves as a *new* Brownian motion under a new measure and the filtration

Then we can rewrite the filtered return dynamics as

$$dR(t) = \bar{\mu}(t)dt + \sigma d\widehat{W}_R(t). \tag{6}$$

The insurer's *posterior* belief on the current regime follows the relationship of

$$dp(t) = [\lambda_L - (\lambda_H + \lambda_L)p(t)]dt + \frac{\mu_H - \mu_L}{\sigma} p(t)(1 - p(t))d\widehat{W}_R(t).^7 \tag{7}$$

Note that the insurer's belief $p(t)$ and the risky asset prices $R(t)$ are perfectly correlated under the PI model, implying that the financial market under the PI model is *complete* from the insurer's point of view. In contrast, the market under the CI model is *incomplete* because there is no financial vehicle for hedging the regime risk.

One advantage of incorporating the non-linear filtering theory into the model is that we can consider a positive learning effect between the realized and expected risky returns. Specifically, the covariance between realized risky returns and the revision in expected risky returns must be

$$Cov(dR(t), d\bar{\mu}(t)) = (\mu_H - \mu_L)^2 p(t)(1 - p(t)) \geq 0, \quad \text{for } p(t) \in [0, 1].$$

The learning effect in the PI model is a concave function with respect to the prior belief, whereas it does not exist under the CI model.[8]

4.2. The Insurer's Optimal Strategies

The insurer's problem under the CI model is to find the value function $V(t, x, p)$:

$$V(t, x, p) = \sup_{\varepsilon, \pi} E^{t,x,p}[U(X(T))]$$

subject to

$$dX(t) = [rX(t) + (\bar{\mu}(t) - r)\pi(t) + (\theta - \eta\varepsilon(t))\alpha]dt + \beta(1 - \varepsilon(t))dW_C(t) + \sigma\pi(t)d\widehat{W}_R(t),$$

$\mathcal{F}^{\widehat{W}_R}(t)$ generated by $\widehat{W}_R(t)$ is equivalent to the filtration $\mathcal{F}^R(t)$. See Theorem 9.1 of Liptser and Shiryaev (2001) for the details.

[7] David (1997) shows that Equation (7) satisfies Lipshitz and growth conditions and, thus, a unique solution exists although it is hard to obtain the explicit distribution for $p(t)$. More properties regarding the posterior belief can be obtained from David (1997), Honda (2003), and Liu (2011).

[8] The dependence on the prior belief contrasts with Zhang et al. (2012), who consider the case where the learning effect does not change over time. See pp. 203 of Zhang et al. (2012).

where $E^{t,x,p}[\cdot]$ is the expectation conditioned on $X(t) = x$ and prior belief $p(t) \equiv p$.

The HJB equation is

$$\begin{aligned}
&V_t + \sup_{\varepsilon,\pi} \Big[\{rx + (\bar{\mu} - r)\pi + (\theta - \eta\varepsilon)\alpha\}V_x + \frac{1}{2}\{\beta^2(1-\varepsilon)^2 + \sigma^2\pi^2 + 2\rho\beta(1-\varepsilon)\sigma\pi\}V_{xx} \\
&+\{\lambda_L - (\lambda_H + \lambda_L)p\}V_p + \frac{(\mu_H - \mu_L)^2}{2\sigma^2}p^2(1-p)^2V_{pp} + \{(\mu_H - \mu_L)\pi p(1-p) \\
&+\rho\beta(1-\varepsilon)\frac{\mu_H - \mu_L}{\sigma}p(1-p)\}V_{xp}\Big] = 0
\end{aligned} \tag{8}$$

with the terminal condition

$$V(T, X(T), p(T)) = U(X(T)).$$

Here, V_t, V_x, and V_p are respectively first–order derivatives with respect to time t, wealth x, and belief p; V_{xx} and V_{pp} are respectively second–order derivatives with respect to wealth x and belief p; and $V_{xp} = \partial^2 V/\partial x \partial p$.

Clearly, variation in posterior beliefs related to changes in future investment opportunities can affect the insurer's decision. In particular, V_p captures the marginal effect of mean–reversion updating, V_{pp} captures the marginal effect of stochastic–belief updating on her value function, and V_{xp} reflects the marginal effect of the covariance between her wealth and belief dynamics. The first–order conditions give us the optimal strategies:

$$\varepsilon^* = \Big[\frac{\alpha\sigma\eta - \rho\beta(\bar{\mu} - r)}{\beta^2\sigma(1-\rho^2)}\Big]\frac{V_x}{V_{xx}} + 1, \quad \text{and} \quad \pi^* = \Big[\frac{\rho\alpha\sigma\eta - \beta(\bar{\mu} - r)}{\beta\sigma^2(1-\rho^2)}\Big]\frac{V_x}{V_{xx}} - \frac{\mu_H - \mu_L}{\sigma^2}p(1-p)\frac{V_{xp}}{V_{xx}}. \tag{9}$$

By following the standard arguments (see, e.g., Zariphopoulou (2001)), We conjecture the value function as

$$V(t, x, p) = -\frac{1}{\gamma}e^{-\gamma c(t)(x + f(t,p))}$$

with the terminal conditions

$$c(T) = 1 \quad \text{and} \quad f(T, p(T)) = 0.$$

Theorem 2. *Let $f_t \equiv \partial f/\partial t$, $f_p \equiv \partial f/\partial p$, and $f_{pp} \equiv \partial^2 f/\partial p^2$,*[9] *Then,*

$$f_t(t,p) - rf(t,p) + \widetilde{\mu}(t,p)f_p(t,p) + \frac{1}{2}\widetilde{\sigma}(t,p)f_{pp}(t,p) + h(t,p) = 0, \tag{10}$$

[9] We assume that f is twice continuously differentiable with respect to $p \in [0,1]$ and one–time continuously differentiable with respect to $t \in [0,T]$.

where

$$\widetilde{\mu}(t,p) = \{\lambda_L - (\lambda_L + \lambda_H)p\} - \frac{\bar{\mu} - r}{\sigma^2}(\mu_H - \mu_L)p(1-p),$$
$$\widetilde{\sigma}(t,p) = \left(\frac{\mu_H - \mu_L}{\sigma}\right)^2 p^2(1-p)^2,$$
$$h(t,p) = \alpha(\theta - \eta) + \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\left[\frac{(\bar{\mu} - r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\bar{\mu} - r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\right].$$

The solution f exists, and the two boundary conditions hold:

$$f_t(t,0) - rf(t,0) + \lambda_L f_p(t,0) + h(t,0) = 0,$$
$$f_t(t,1) - rf(t,1) - \lambda_H f_p(t,1) + h(t,1) = 0.^{10}$$

Sketch of the Proof. The proof of the existence of the solution f coincides with that of the solution $V(t,x,p)$ to the HJB equation.[11] We can show that derivatives such as $f_p(t,p)$, $f_{pp}(t,p)$, and $f_t(t,p)$ are bounded and sufficiently differentiable (See Proposition 2 of Honda (2003) in particular). In detail, we provide the derivation of Equation (10) in Appendix B. □

Theorem 2 shows that the level of $f(t,p)$ represents the insurer's belief adjustment of the reinsurance and investment strategies concerning variation in posterior beliefs. This argument follows from the fact that the belief adjustment affects the conjectured value function V only through $f(t,p)$.[12] In short, we call $f(t,p)$ *certainty equivalent wealth* (CEW) that captures the wealth level of variation in posterior beliefs.

4.3. The Properties of the Certainty Equivalent Wealth f

The insurer in the PI model can update her belief on the current market condition (or on the information about current regime) by reflecting the market information. Thus, we can obtain the effect of *information quality* on the insurer's optimal reinsurance and investment behaviors by comparing f_is of the CI model with f of the PI model. To this end, we first examine the properties of the CEW f.

Define the conditional *market price of risk*, $\vartheta(t)$, under the PI model as

$$\vartheta(t) \equiv \frac{\bar{\mu}(t) - r}{\sigma} = \frac{\mu_L + (\mu_H - \mu_L)p(t) - r}{\sigma}.$$

[11] The PDE has a degenerate form near the boundaries at $p = 0$ or 1. Thus, the potential degenerate PDE enables us to require whether f is sufficiently smooth enough to be differentiable.
[12] We can easily get f by exploiting a numerical method.

Note that $\vartheta(t)$ is within the interval of $[(\mu_L - r)/\sigma, (\mu_H - r)/\sigma]$.[13] Thus, there exists an equivalent Martingale measure $\widetilde{\mathbb{P}}$ with respect to the original measure $\mathbb{P}$ given by $d\widetilde{\mathbb{P}}/d\mathbb{P} = Z(T)$[14] for

$$Z(t) = \exp\left\{ -\int_0^t \vartheta(s) d\widetilde{W}_R(s) - \frac{1}{2}\int_0^t \vartheta^2(s) ds \right\}, \quad \text{with} \quad \widetilde{W}_R(t) = \widehat{W}_R(t) + \int_0^t \vartheta(s) ds. \tag{11}$$

Proposition 1. *The CEW $f(t,p)$ has the Feynman–Kac representation:*[15]

$$\begin{aligned} f(t,p) = & \ \widetilde{E}^{t,p}\left[\int_t^T e^{-r(s-t)} h(s,p(s)) ds\right] \\ = & \ \frac{\alpha(\theta-\eta)}{r}\left[1 - e^{-r(T-t)}\right] + \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)} \widetilde{E}^{t,p}\left[\int_t^T \frac{1}{2}\vartheta(s)^2 - \rho \cdot \frac{\alpha\eta}{\beta}\vartheta(s) + \frac{\alpha^2\eta^2}{2\beta^2} ds\right], \end{aligned}$$

where $\widetilde{E}^{t,p}[\cdot]$ is an expectation conditioned on the time-t condition $p(t) = p$ under $\widetilde{\mathbb{P}}$.

Proof: The sketch of the proof is in Appendix C. □

Proposition 1 implies the CEW f can be decomposed into two components:

- The CEW by reinsurance costs, $f^C(t)$:

$$f^C(t) \equiv \frac{\alpha(\theta-\eta)}{r}\left[1 - e^{-r(T-t)}\right],$$

which depends on both time to maturity $(T-t)$ and the difference between the safety loadings, $\theta - \eta \leq 0$. Note that $f^C \leq 0$, implying the insurer perceives it as her capital losses.

- The CEW by demands on precautionary saving, $f^S(t,p)$:

$$f^S(t,p) \equiv \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)} \widetilde{E}^{t,p}\left[\int_t^T \frac{1}{2}\vartheta(s)^2 - \rho \cdot \frac{\alpha\eta}{\beta}\vartheta(s) + \frac{\alpha^2\eta^2}{2\beta^2} ds\right].$$

Since $(\alpha\eta/\beta)^2(\rho^2 - 1) \leq 0$, the second component f^S driven by demands on precautionary saving against both market risk and insurance claims risk has a non-negative value. Obviously, it has a small value for high risk aversion γ.

[13] Obviously, the bounded market price of risk satisfies Novikov's condition (the Dominated Convergence Theorem). Moreover, the bounded condition makes the wealth dynamics in the PI model to satisfy the Lipschitz and growth conditions, which must be met to justify use of the standard verification theorem (Dybvig et al., 1999).

[14] See Proposition 5.B of Duffie (2001). Here, $Z(t)$ is a positive $\widetilde{\mathbb{P}}$–Martingale due to $Z(0) = 1$.

[15] See Theorem 7.6 of Karatzas and Shreve (1991).

Subsequently, the positivity of the CEW f is dependent on the dominance of the CEW over demands on precautionary saving against the CEW by reinsurance costs, i.e., $f^S > f^C$.

In addition, the CEW f goes to infinity as the stock prices and insurance claims are perfectly (positively or negatively) correlated, i.e., $\lim_{|\rho|\to 1} f(t,p) = \infty$. The correlation can be regarded as a measure of how the claims risk is easily shared by the financial market. Thus, the insurer with insurance claims which is highly correlated with the market movement has high motive to exploit the *risk–sharing effect*.

Proposition 2. *The first–order marginal CEW $f_p(t,p) > 0$ if, for all $t \in [0,T]$, $\vartheta(t) > \rho\alpha\eta/\beta$ holds. Moreover, the second–order marginal CEW $f_{pp}(t,p)$ can have both positive and negative values.*

Proof: Applying the Malliavin calculus (Honda, 2003), The boundedness and differentiablity of $f(t,p)$ (see the Sketch of the Proof in Theorem 2) yield

$$\begin{aligned} f_p(t,p) = & \ \widetilde{E}^{t,p}\Big[\int_t^T e^{-r(s-t)} \frac{\partial h(s,p(s))}{\partial p(s)} \frac{\partial p(s)}{\partial p} ds\Big] \\ = & \ \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)} \frac{\mu_H - \mu_L}{\sigma} \widetilde{E}^{t,p}\left[\int_t^T \left\{\vartheta(s) - \rho\cdot\frac{\alpha\eta}{\beta}\right\} I(s) ds\right] > 0, \end{aligned}$$

where $I(s) \equiv \partial p(s)/\partial p$ with $I(0) = 1$. We present a stochastic differential equation form of $I(t)$ in Appendix D. Moreover, f can be either convex or concave with respect to p, depending on the value of f_{pp}. Applying the Malliavin calculus again, we get

$$\begin{aligned} f_{pp}(t,p) = & \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)} \left(\frac{\mu_H - \mu_L}{\sigma}\right)^2 \widetilde{E}^{t,p}\left[\int_t^T I(s)^2 ds\right] \\ & + \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)} \frac{\mu_H - \mu_L}{\sigma} \widetilde{E}^{t,p}\left[\int_t^T \left\{\vartheta(s) - \rho\cdot\frac{\alpha\eta}{\beta}\right\} J(s) ds\right], \end{aligned}$$

where $J(t) \equiv \partial I(t)/\partial p$ with $J(0) = 0$. Since $J(0) = 0$, $J(t)$ can be positive or negative. □

The first statement of Proposition 2 implies that a sufficiently high expected risky return in Low regime, μ_L, guarantees the increasing property of the CEW f with the initial belief p.[16] The second statement of Proposition 2 implies that the CEW f can be a convex or concave function

[16] Strictly speaking, $\mu_L > r + \rho\cdot\frac{\alpha\eta}{\beta}\sigma$.

with respect to the initial belief p according to the parameters condition.

5. Implications

5.1. Parameters

We use parameters in Jang and Kim (2015), who solve the ruin minimization problem for an insurer under the two-state observable Markov chain. Jang and Kim (2015) estimated market parameters and insurance claim parameters by utilizing KOSPI data and Korean property and casualty insurance market data. In fact, they estimated the parameters in the two different models, and we choose the parameters of Model 1 which restricts only the drift of insurance claims process as a constant value across the two regimes. We set the other parameters by utilizing the relationship of

$$\text{Parameter}_{ave} = \frac{\lambda_L}{\lambda_L + \lambda_H}\text{Parameter}_H + \frac{\lambda_H}{\lambda_L + \lambda_H}\text{Parameter}_L.$$

Note that $\frac{\lambda_L}{\lambda_L+\lambda_H}$ is the expected duration of staying in regime H, and $\frac{\lambda_H}{\lambda_L+\lambda_H}$ is that in regime L.

Specifically, the benchmark parameters are $\mu_H = 0.1188$, $\mu_L = -0.2592$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, and $T = 5$.[17] We set the baseline coefficient of absolute risk aversion as $\gamma = 20$.

5.2. Optimal Reinsurance Strategy

In this section, we compare the optimal reinsurance rates ε^* (the PI model) with ε_i^* for $i \in \{H, L\}$ (the CI model). Specifically, we rewrite ε^* and ε_i^* in terms of two risk–adjusted premiums between the two markets and then study the corresponding implications.

[Insert Figure 1 here.]

The mark–to–market reinsurance policy ε^* (Figure 1) depends on prior belief p. This argument can be clearly justified by the existence of the market price of risk $\vartheta(0) \equiv \frac{\mu_L+(\mu_H-\mu_L)p-r}{\sigma}$:

$$\varepsilon^* = 1 + \frac{e^{-rT}}{\gamma\beta(1-\rho^2)}\left[\rho \cdot \vartheta(0) - \frac{\alpha\eta}{\beta}\right].$$

[17] The sign of ρ differs from that presented in Jang and Kim (2015). This difference results from the negative relation of the claims process $dC(t)$ to claims shock $dW_C(t)$ (See footnote 14 in Jang and Kim (2015)).

In addition, ε^* is a linearly decreasing function of prior belief (Figure 1) with slope

$$\frac{e^{-rT}}{\gamma\beta(1-\rho^2)}\frac{\mu_H-\mu_L}{\sigma}\cdot\rho<0$$

as a result of $\rho<0$.

This negative slope suggests a *counter–cyclical reinsurance mechanism*. For example, the insurer's pessimistic view $p\to 0$ close to regime L induces her to further raise the reinsurance rates to maximize her terminal wealth. Indeed, the counter–cyclical mechanism can help prevent considerable losses to meet liability obligations, especially during the recent financial crisis (CGFS, 2011).[18]

In contrast, ε_i^* for $i\in\{H,L\}$ is independent of prior belief:

$$\varepsilon_i^*=1+\frac{e^{-rT}}{\gamma\beta(1-\rho^2)}\left[\rho\cdot\vartheta_i-\frac{\alpha\eta}{\beta}\right],$$

where ϑ_H represents the best investment opportunities, but ϑ_L represents the worst investment opportunities. In short, ε_i^* only provides the quantitative reinsurance ceiling and floor; ε_L^* is the upper limit of ε^*, and ε_H^* is the lower limit such that $\varepsilon^*\in[\varepsilon_H^*,\varepsilon_L^*]$ because $\rho<0$.

As a matter of fact, the use of the low correlation $\rho=-0.0222$ makes the slope of ε^* almost flat. It seems to be independent of prior belief (Figure 1). However, this small effect on the mark–to–market policy ε^* may be confined into Korean markets, in which policy differs from those of other countries (OECD, 2015). In the United States, for example, the insurance sector is highly correlated with the banking sector (CGFS, 2011). This high correlation can result in substantial variation in the counter–cyclical reinsurance mechanism.

The relative importance of the (correlated) market price of risk $\rho\cdot\vartheta(0)$ and the reinsurance price of risk $\frac{\alpha\eta}{\beta}$ significantly affects the counter–cyclical reinsurance policy ε^*. For example, the condition $\rho\cdot\vartheta(0)=\frac{\alpha\eta}{\beta}$ corresponds to $\varepsilon^*=1$ so that she wants to perfectly hedge the claims risk with the resulting prior:

$$p_{\varepsilon^*}\equiv\frac{\sigma}{\rho(\mu_H-\mu_L)}\left[\frac{\alpha\eta}{\beta}-\rho\cdot\vartheta_L\right].$$

The insurer' perfect–hedge desire to be $\varepsilon^*=1$ occurs when she compares the counter–cyclical worst case $\rho\cdot\vartheta_L$ with her target reinsurance opportunities $\frac{\alpha\eta}{\beta}$. Consequently, we find $\rho\cdot\vartheta(0)<\frac{\alpha\eta}{\beta}$ for

[18] CGFS is the abbreviation of "the Committee on the Global Financial System".

all $p \in [0,1]$; thus, we can exclude the possibility that $\varepsilon^* \geq 1$.

In summary, both ε^* and ε_i^* are myopic decisions. The myopic decisions result from her information confidence, *not* from variation in posterior beliefs. However, insurance companies that are eager to monitor a financial market can adopt the mark–to–market reinsurance strategy that moves on a counter–cyclical basis.

5.3. Optimal Portfolio Strategy

In this section, we compare the optimal portfolio strategy π^* (the PI model) with π_i^* for $i \in \{H, L\}$ (the CI model). Specifically, we restate π^* and π_i^* in terms of the two risk–adjusted premiums and then study the corresponding implications.

The mark–to–market investment strategy π^* depends on prior belief p. We can further decompose π^* into a *myopic demand* π_1^* and an *intertemporal hedging demand* π_2^*:

$$\pi^* = \underbrace{\frac{e^{-rT}}{\gamma\sigma(1-\rho^2)}\left[\vartheta(0) - \rho \cdot \frac{\alpha\eta}{\beta}\right]}_{\pi_1^*} + \underbrace{\frac{\mu_H - \mu_L}{\sigma^2} p(1-p) \cdot -f_p(0,p)}_{\pi_2^*}.$$

The non–myopic decision π_2^* is attributed to her marginal CEW f_p that results from variation in posterior beliefs.

In contrast, π_i^* for $i \in \{H, L\}$ only provides the quantitative investment ceiling and floor:

$$\pi_i^* = \frac{e^{-rT}}{\gamma\sigma(1-\rho^2)}\left[\vartheta_i - \rho \cdot \frac{\alpha\eta}{\beta}\right].$$

That is, π_H^* can be the upper bound of π^* at $p = 1$ and π_L^* is the lower bound at $p = 0$ without the intertemporal hedging demand π_2^* such that $\pi^* \in [\pi_L^*, \pi_H^*]$.

This simple difference in information quality is consistent with the conventional results that future uncertainty in relation to estimation risk generates an intertemporal hedging demand (Campbell et al., 2004; Chacko and Viceira, 2005). Our analysis also supports the argument of Ang and Bekaert (2002), who show that when an economic agent is uncertain about macroeconomic conditions, the regime effect weakens by the hedging demand.

[Insert Figure 2 here.]

Specifically, we attempt to analyze the mark–to–market investment policy π^* and the CEW f related to variation in posterior beliefs with two cases: one in which short sales are allowed (Figure 2), and one in which short sales are not allowed (Figure 3).

First, the short–sale case originates from the condition $\mu_L < r$ (Section 5.1). The insurer can increase her risk–free profit in regime L, so she has no incentive to invest in risky assets. Consequently, the availability of the worst financial–market opportunities makes short sales optimal, although it is not legally allowed for insurance companies in most countries (OECD, 2015). Thus, we will conduct a study of whether limiting short sales affects both her mark–to–market asset allocation π^* and the CEW f.

Second, a simple restriction $\mu_L = r$ can yield no short–sale case. This restriction satisfies $\mu_L > r + \rho \cdot \frac{\alpha\eta}{\beta}\sigma$ due to $\rho < 0$ and also coincides with

$$\vartheta(t) > \rho \cdot \frac{\alpha\eta}{\beta}$$

for all $t \in [0, T]$ (Proposition 2). Likewise, the restriction also affects the mark–to–market reinsurance policy ε^*. However, it only changes the slope variation (Section 5.2), *not* the general pattern.

The myopic demand $\pi_1^* \equiv \frac{e^{-rT}}{\gamma\sigma(1-\rho^2)}\left[\vartheta(0) - \rho \cdot \frac{\alpha\eta}{\beta}\right]$ is a linearly increasing function of prior belief p with slope:

$$\frac{e^{-rT}}{\gamma\sigma(1-\rho^2)} \frac{\mu_H - \mu_L}{\sigma} > 0.$$

Thus, the myopic demand π_1^* corresponds to a *pro–cyclical investment mechanism*: the insurer's optimistic view $p \to 1$ close to regime H increases the amount invested.

The relative importance between the (correlated) reinsurance price of risk $\rho \cdot \frac{\alpha\eta}{\beta}$ and the current market price of risk $\vartheta(0)$ significantly affects the pro–cyclical mechanism. For example, the condition $\vartheta(0) = \rho \cdot \frac{\alpha\eta}{\beta}$ corresponds to $\pi_1^* = 0$ so that she has no pro–cyclical demand. The resulting break–even prior to hold $\vartheta(0) = \rho \cdot \frac{\alpha\eta}{\beta}$ delivers

$$p_{\pi_1^*} \equiv \frac{\sigma}{\mu_H - \mu_L}\left[\rho \cdot \frac{\alpha\eta}{\beta} - \vartheta_L\right].$$

Her desire to eliminate the pro–cyclical demand arises from the comparison between the reinsurance opportunities $\rho \cdot \frac{\alpha\eta}{\beta}$ and the worst investment opportunity set ϑ_L. Moreover, the condition $\vartheta(0) < \rho \cdot \frac{\alpha\eta}{\beta}$ results in $\pi_1^* < 0$ implies that short sales occur when the current investment opportunities are not preferable to the reinsurance opportunities for her risk management. Thus, good market signal $\vartheta(0)$ relative to $\rho \cdot \frac{\alpha\eta}{\beta}$ gives incentive to increase the pro–cyclical portfolio amount, i.e., $\vartheta(0) > \rho \cdot \frac{\alpha\eta}{\beta}$.

[Insert Figure 3 here.]

In the short–sale case (Panel A, Figure 2), π_1^* is negative at $p < 0.70$ but positive at $p > 0.70$. This conclusion indicates that $p_{\pi_1^*} = 0.70$. In the no short–sale case (Panel A, Figure 3), however, π_1^* is always positive as a result of $\vartheta(0) > \rho \cdot \frac{\alpha\eta}{\beta}$ for all $p \in [0, 1]$, so no break–even prior $p_{\pi_1}^*$ exists.

The hedging demand $\pi_2^* \equiv \frac{\mu_H - \mu_L}{\sigma^2} p(1-p) \cdot -f_p$ is a non–linear function of prior belief. This demand functions as a *counter–cyclical investment mechanism* to offset the potential overreaction to the pro–cyclical mechanism:

$$\pi_2^* = \frac{e^{-rT}}{\gamma\sigma(1-\rho^2)} \left(\frac{\mu_H - \mu_L}{\sigma}\right)^2 p(1-p) \cdot \widetilde{E}^{0,p} \left[\int_0^T \left\{\rho \cdot \frac{\alpha\eta}{\beta} - \vartheta(s)\right\} I(s) ds\right],$$

where

$$f_p(0, p) \equiv \frac{e^{-rT}}{\gamma(1-\rho^2)} \frac{\mu_H - \mu_L}{\sigma} \cdot \widetilde{E}^{0,p} \left[\int_0^T \left\{\vartheta(s) - \rho \cdot \frac{\alpha\eta}{\beta}\right\} I(s) ds\right].$$

In short, $\rho \cdot \frac{\alpha\eta}{\beta} - \vartheta(t)$ for all $t \in [0, T]$ in π_2^* represents the counter–cyclical mechanism, whereas $\vartheta(0) - \rho \cdot \frac{\alpha\eta}{\beta}$ in π_1^* represents the pro–cyclical mechanism.

Now, we relate the counter–cyclical mechanism to the CEW f. In particular, we must study (i) the first–order marginal CEW f_p that determines either the increasing or decreasing property for the CEW f, and (ii) her second–order marginal CEW f_{pp} that determines whether f is convex or concave.

As a consequence, we find that the CEW f, which reflects the belief adjustment of reinsurance and investment strategies, must be convex with respect to prior belief. This convex property can be justified by our numerical results that f_{pp} is strictly positive for all $p \in [0, 1]$, regardless of short–sale allowance. Further, the CEW decomposition $f(t, p) = f^C(t) + f^S(t, p)$ shows that the CEW by demands on precautionary saving, f^S, is also convex with the same property as f, because the CEW by reinsurance costs, f^C, is independent of prior belief (Proposition 1). Let us go into further details.

In the short–sale case (Panel B, Figure 2), π_2^* is positive at $p < 0.61$ but negative at $p > 0.61$. Given $\pi_2^* \equiv \frac{\mu_H - \mu_L}{\sigma^2} p(1-p) \cdot -f_p$ and $\frac{\mu_H - \mu_L}{\sigma^2} p(1-p) \geq 0$, her marginal behavior f_p must be negative at $p < 0.61$ but positive at $p > 0.61$. The fact $f_p < 0$ at $p < 0.61$ implies that f is decreasing with respect to prior belief and vice versa.

The counter–cyclical break–even prior $p_{\pi_2^*}$ defined to be $f_p = 0$ suggests that she feels change in preference between the two risk–adjusted premiums at $p_{\pi_2^*} = 0.61$. For example, the fact that f_p is positive at $p > 0.61$ implies that the current investment opportunities are more favorable than the (correlated) reinsurance opportunities. This relative importance reflects the demands on precautionary saving, which functions as the counter–cyclical mechanism as a result of the negative effect of $f_p = f_p^S$ on π_2^*.

In the no short–sale case (Panel B, Figure 3), π_2^* is consistently negative. This result implies that the counter–cyclical demand always has a negative effect on the total portfolio demand π^*. The central difference in the short–sale case (Figure 2) arises from the consistent predominance of $\vartheta(t)$ over $\rho \cdot \frac{\alpha\eta}{\beta}$ for all $t \in [0, T]$. The fact $\vartheta(t) > \rho \cdot \frac{\alpha\eta}{\beta}$ for all $t \in [0, T]$ indicates that the marginal CEW f_p is always positive.

In addition, the hedging demand π_2^* depends on her risk aversion γ and the correlation ρ that also determine the precautionary saving's component f^S. First, an increase in γ decreases the magnitude of π_2^* (Panels B, Figure 2 and Figure 3). This negative relationship implies that higher risk aversion decreases her wealth level due to the high demands on precautionary saving:

$$\lim_{\gamma\to\infty} f_p(t,p) = \lim_{\gamma\to\infty} \frac{\partial f^S(t,p)}{\partial p} = 0.$$

Second, an increase in ρ increases the magnitude of π_2^*. This positive relationship means that as the risk–sharing effect increases so that the demands on precautionary saving decrease, her wealth also increases as

$$\lim_{|\rho|\to 1} f_p(t,p) = \lim_{|\rho|\to 1} \frac{\partial f^S(t,p)}{\partial p} = \infty.$$

The total portfolio demand $\pi^* = \pi_1^* + \pi_2^*$ seems to deviate *little* from the pro–cyclical demand π_1^* (Panels C, Figure 2 and Figure 3). The small size of this deviation may result from the low correlation $\rho = -0.022$ in Korean markets. We believe that other countries in which the risk–sharing effect is higher than this will exhibit substantial counter–cyclical behavior (CGFS, 2011).

Actually, many insurance companies suffered substantial losses on their portfolios during the recent financial crisis (OECD, 2015). These losses stimulated the development of the counter–cyclical investment mechanism. Our counter–cyclical implications can help protect against serious losses and can also improve the financial stability of these companies.

5.4. The Effect of Information Quality

Comparing the CI model with the PI model in terms of certainty equivalent wealth can lead us to get economic implication concerning the effect of information quality on the insurer's optimal strategies.

Definition 1. *The certainty equivalent wealth gain by information acquisition (I-CEWG),* $\Delta_i(\cdot)$*,* *in each regime $i \in \{H, L\}$ is defined as*

$$V_i(t, x - \Delta_i(p)) = V(t, x, p). \tag{12}$$

We also define the regime–weighted average $\Delta_{ave}(p)$ (Jang et al., 2007):

$$\Delta_{ave}(p) = \frac{\lambda_L}{\lambda_L + \lambda_H}\Delta_1(p) + \frac{\lambda_H}{\lambda_L + \lambda_H}\Delta_2(p).$$

We plot the three kinds of the I-CEWGs regarding information quality with $\gamma = 20$ and $x = 0.5$: the short–sale case $\mu_L < r$ (Panel A, Figure 4), and no short–sale case with the restriction $\mu_L = r$ (Panel B, Figure 4). The main findings are as follows.

[Insert Figure 4 here.]

First, information quality certainly exists as a result of the positive $\Delta_i(p)$ for $i \in \{H, L, ave\}$ (Figure 4). To see this argument rigorously, we specify the exponents in the value functions in Equation (12):

$$f_i(0) - \Delta_i(p) = f(0, p).$$

The strictly positive $\Delta_i(p) = f_i(0) - f(0, p)$ implies that the constant $f_i(0)$ independent of prior belief p is strictly larger than the variable $f(0, p)$. Therefore, f leads to smaller variation in wealth than does f_is based on high information confidence. In this regard, the small effect on wealth corresponds to conservative risk management strategy that arises from changes in future investment opportunities.

Second, information quality is a concave function of prior belief. This conclusion can be easily inferred from the characteristic of f determined by the relative importance of the risk–adjusted premiums (Section 5.3): the CEW f is a convex function of prior belief, regardless of whether to allow short sales. Thus, the positive $\Delta_i(p) = f_i(0) - f(0, p)$ naturally corresponds to the concavity.

Third, information quality is affected by whether short sales are limited. In the short–sale case (Panel A), $\Delta_L(p)$ is larger than $\Delta_H(p)$, implying $f_L(0) > f_H(0)$. The reason why $f_L(0)$ is larger than $f_H(0)$ is that short sales allow her even to avail bad investment opportunities to maximize her wealth, which is associated with $|\pi_L^*| > |\pi_H^*|$ (Panel C, Figure 2). In the no short–sale case (Panel B), however, $\Delta_H(p)$ is larger than $\Delta_L(p)$, leading to $f_H(0) > f_L(0)$. When short sales are not allowed, she only avails good investment opportunities, which is related to $|\pi_H^*| > |\pi_L^*|$ (Panel C, Figure 3).

6. Conclusion

This paper investigates the optimal mark–to–market reinsurance and asset investment strategies for insurers with complete or partial information on expected return. The insurer with partial information is assumed to have prior belief on the expected return and to update her posterior beliefs by exploiting its price information. We show that the strategies of the insurer with partial information can be highly dependent on prior belief, and that variation in posterior beliefs gives rise to her counter–cyclical investment demand. By comparing the two insurers' strategies, we show that insurer's utility gain by the information acquisition is a concave function with respect to prior belief. This conclusion can be explained by the relative importance between reinsurance costs and demand on precautionary saving.

Appendix A. The Proof of Theorem 1

Fist of all, we have the two terminal conditions $c(T) = 1$ and $f_i(T) = 0$.

Plugging Equation (4) into Equation (3), we get the HJB equations for each regime $i \in \{H, L\}$:

$$\begin{aligned} V_{i,t} + \{rx + \alpha(\theta - \eta)\}V_{i,x} - \frac{1}{1-\rho^2}\left[\frac{(\mu_i - r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\mu_i - r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\right]\frac{V_{i,x}^2}{V_{i,xx}} \\ + \lambda_i\{V_j - V_i\} = 0. \end{aligned} \tag{A.1}$$

The first–order and second–order derivatives are

$$\begin{aligned} V_{i,t} &= -\tfrac{1}{\gamma}[-\gamma c_t(x + f_i) - \gamma c f_{i,t}]e^{-\gamma c(x+f_i)}, \\ V_{i,x} &= c e^{-\gamma c(t)(x+f_i)}, \\ V_{i,xx} &= -\gamma c^2 e^{-\gamma c(t)(x+f_i)}, \end{aligned}$$

where $c_t \equiv df\partial c\partial t$ and $f_{i,t} \equiv \dfrac{\partial f_i}{\partial t}$. Rearranging Equation (A.1) gives

$$\begin{aligned} &\Big[c_t + rc\Big]x + c\Big[f_{i,t} + \frac{c_t}{c}f_i + \frac{\lambda_i}{\gamma c}\left\{1 - e^{-\gamma c(f_j - f_i)}\right\} \\ &+ \alpha(\theta - \eta) + \frac{1}{\gamma c(1-\rho^2)}\left\{\frac{(\mu_i - r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\mu_i - r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\right\}\Big] = 0. \end{aligned}$$

To obtain the solution independent of x, the terms in the two square brackets must be zero. Then Equation (5) in Theorem 1 is straightforward.

The reader can easily provide a verification theorem for V if he/she can just follow the arguments in Jang et al. (2007).

Appendix B. The Proof of Theorem 2

Plugging Equation (9) into Equation (8) corresponds to the HJB equation:

$$\begin{aligned} V_t + (rx + \alpha\theta - \alpha\eta)V_x + \frac{1}{1-\rho^2}\Big[\frac{\alpha\eta\rho(\bar{\mu}-r)}{\beta\sigma} - \frac{(\bar{\mu}-r)^2}{2\sigma^2} - \frac{\alpha^2\eta^2}{2\beta^2}\Big]\frac{V_x^2}{V_{xx}} \\ -\frac{\bar{\mu}-r}{\sigma^2}(\mu_H-\mu_L)p(1-p)\frac{V_xV_{xp}}{V_{xx}} - \frac{(\mu_H-\mu_L)_\mu^2}{2\sigma^2}p^2(1-p)^2\frac{V_{xp}^2}{V_{xx}} \\ +\{\lambda_L - (\lambda_H+\lambda_L)p\}V_p + \frac{(\mu_H-\mu_L)^2}{2\sigma^2}p^2(1-p)^2V_{pp} = 0. \end{aligned} \tag{B.1}$$

Recall that the conjecture value function $V(t,p,x) = -\frac{1}{\gamma}e^{-\gamma c(t)(x+f(t,p))}$ with the terminal conditions $c(T)=1$ and $f(T,p(T))=0$. The first–order and second–order derivatives are

$$\begin{aligned} &V_t = -\gamma[c_t(x+f)+cf_t]V, \qquad V_x = -\gamma cV, \qquad V_{xx} = \gamma^2c^2V, \\ &V_p = -\gamma cf_pV, \qquad V_{pp} = -\gamma cf_{pp}V + \gamma^2c^2f_p^2V, \qquad V_{xp} = \gamma^2c^2f_pV. \end{aligned}$$

Substituting the derivatives into Equation (B.1) yields

$$\Big[c_t + rc\Big]x + c\Big[f_t + \frac{c_t}{c}f + \widetilde{\mu}_p(t,p)f_p + \frac{1}{2}\widetilde{\sigma}_p(t,p)f_{pp} + h\Big] = 0,$$

where

$$\begin{aligned} \widetilde{\mu}_p(t,p) &= \{\lambda_L - (\lambda_L+\lambda_H)p\} - \frac{\bar{\mu}-r}{\sigma^2}(\mu_H-\mu_L)p(1-p), \\ \widetilde{\sigma}_p(t,p) &= \left(\frac{\mu_H-\mu_L}{\sigma}\right)^2 p^2(1-p)^2, \\ h(t,p) &= \alpha(\theta-\eta) + \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\left[\frac{(\bar{\mu}-r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\bar{\mu}-r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\right]. \end{aligned}$$

The first square bracket with $c(T)=1$ leads to $c(t)=e^{r(T-t)}$, and the second square bracket with $f(T,p(T))=0$ delivers the following linear second–order PDE:

$$f_t(t,p) - rf(t,p) + \widetilde{\mu}_p(t,p)f_p(t,p) + \frac{1}{2}\widetilde{\sigma}_p(t,p)f_{pp}(t,p) + h(t,p) = 0.$$

Now, we can provide the verification theorem for the PI model. In fact, the standard arguments can be applied (Dybvig et al., 1999), and it is almost a copy of Proposition 1 in Honda (2003).

Appendix C. Proof of Proposition 1

By the Girsanov theorem in Equation (11), we can rewrite Equations (6) and (7) as

$$\begin{aligned} dR(t) &= rdt + \sigma d\widetilde{W}_R(t), \\ dp(t) &= \widetilde{\mu}_p(t, p(t))dt + \widetilde{\sigma}_p(t, p(t))d\widetilde{W}_R(t), \end{aligned} \tag{C.1}$$

where the drift and diffusion terms of the belief process under $\widetilde{\mathbb{P}}$ are given by

$$\begin{aligned} \widetilde{\mu}_p(t, p(t)) &= \lambda_L - (\lambda_H + \lambda_L)p(t) - \left(\frac{\mu_H - \mu_L}{\sigma}\right) p(t)(1 - p(t))\vartheta(t), \\ \widetilde{\sigma}_p(t, p(t)) &= \left(\frac{\mu_H - \mu_L}{\sigma}\right) p(t)(1 - p(t)). \end{aligned}$$

David (1997) shows that the original belief process under $\mathbb{P}$ satisfies the Lipschitz and growth conditions; thus, there exists a unique solution. Indeed, the bounded market price of risk $\vartheta(t)$ also suffices that the adjusted belief process under $\widetilde{\mathbb{P}}$ also has a unique solution.

Next, we briefly sketch the proof of Proposition 1. Define a process $Q(s)$ as

$$Q(s) = e^{-r(s-t)} f(s, p(s)) + \int_t^s e^{-r(u-t)} h(u, p(u))du. \tag{C.2}$$

Applying the Itô formula to $Q(s)$ yields

$$\begin{aligned} dQ(s) &= -re^{-r(s-t)} f + e^{-r(s-t)} \left(f_t + \widetilde{\mu}_p f_p + \frac{1}{2}\widetilde{\sigma}_p^2 f_{pp} + h \right) ds + e^{-r(s-t)} \widetilde{\sigma}_p f_p d\widetilde{W}_R(s) \\ &= e^{-r(s-t)} \left(-rf + f_t + \widetilde{\mu}_p f_p + \frac{1}{2}\widetilde{\sigma}_p^2 f_{pp} + h \right) ds + e^{-r(s-t)} \widetilde{\sigma}_p f_p d\widetilde{W}_R(s). \end{aligned}$$

We easily find that the first parenthesis in the drift term is equal to the PDE in Equation (10) and then rewrite the remaining term as

$$dQ(s) = e^{-r(s-t)} \widetilde{\sigma}_p f_p d\widetilde{W}_R(s).$$

The Martingale property is easily obtained by

$$\widetilde{E}^{t,p} \left[Q(T) - Q(t) \right] = \widetilde{E}^{t,p} \left[\int_t^T e^{-r(s-t)} \widetilde{\sigma}_p f_p d\widetilde{W}_R(s) \right] = 0,$$

provided that f_p is bounded on the domain $p(t) \in [0, 1]$. Here, Proposition 2 of Honda (2003) shows that the marginal function f_p is bounded by the Dominated convergence theorem.

It is straightforward to show from Equation (C.2):

$$\begin{aligned}\widetilde{E}^{t,p}[Q(T)] &= \widetilde{E}^{t,p}\left[e^{-r(T-t)}f(T,p(T))\right] + \widetilde{E}^{t,p}\left[\int_t^T e^{-r(s-t)}h(s,p(s))ds\right],\\ \widetilde{E}^{t,p}[Q(t)] &= f(t,p),\end{aligned}$$

with the terminal condition:

$$f(T,p(T)) = 0.$$

This completes the proof:

$$f(t,p) = \widetilde{E}^{t,p}\left[\int_t^T e^{-r(s-t)}h(s,p(s))ds\right], \quad t \in [0,T].$$

Appendix D. Proof of Proposition 2

We start with the belief process under $\widetilde{\mathbb{P}}$ in Equation (C.1). First, we define a process $I(t)$ as

$$I(t) = \frac{\partial}{\partial p}p(t).$$

Then, we can derive the SDE of $I(t)$ by using the the Malliavin calculus:

$$\begin{aligned}dI(t) &= \frac{\partial \widetilde{\mu}_p(t,p(t))}{\partial p(t)}\frac{\partial p(t)}{\partial p}dt + \frac{\partial \widetilde{\sigma}_p(t,p(t))}{\partial p(t)}\frac{\partial p(t)}{\partial p}d\widetilde{W}_R(t),\\ &= \frac{\partial \widetilde{\mu}_p(t,p(t))}{\partial p(t)}I(t)dt + \frac{\partial \widetilde{\sigma}_p(t,p(t))}{\partial p(t)}I(t)d\widetilde{W}_R(t) \quad \text{with} \quad I(0)=1,\end{aligned}$$

where $\frac{\partial \widetilde{\mu}_p(t,p(t))}{\partial p(t)}$ and $\frac{\partial \widetilde{\sigma}_p(t,p(t))}{\partial p(t)}$ are given by

$$\begin{aligned}\frac{\partial \widetilde{\mu}_p(t,p(t))}{\partial p(t)} &= 3\left(\frac{\mu_H-\mu_L}{\sigma}\right)^2 p(t)^2 + 2\left\{\frac{(\mu_L-r)(\mu_H-\mu_L)}{\sigma^2} - \left(\frac{\mu_H-\mu_L}{\sigma}\right)^2\right\}p(t)\\ &\qquad - \left\{\lambda_L+\lambda_H+\frac{(\mu_L-r)(\mu_H-\mu_L)}{\sigma^2}\right\},\\ \frac{\partial \widetilde{\sigma}_p(t,p(t))}{\partial p(t)} &= (1-2p(t))\left(\frac{\mu_H-\mu_L}{\sigma}\right).\end{aligned}$$

Here, $\frac{\partial \widetilde{\mu}_p(t,p(t))}{\partial p(t)}$ and $\frac{\partial \widetilde{\sigma}_p(t,p(t))}{\partial p(t)}$ also satisfy the Lipschitz and growth conditions over the bounded interval $p(t) \in [0,1]$. The fact $I(0)=1$ demonstrates the strictly positive $I(t)$.

Second, we define a process $J(t)$ as

$$J(t) = \frac{\partial}{\partial p} I(t).$$

The SDE of $J(t)$ evolves as

$$\begin{aligned} dJ(t) &= \left(\frac{\partial \widetilde{\mu}_p}{\partial p(t)} \frac{\partial I(t)}{\partial p} + \frac{\partial^2 \widetilde{\mu}_p}{\partial p(t)^2} \left(\frac{\partial p(t)}{\partial p} \right)^2 \right) dt + \left(\frac{\partial \widetilde{\sigma}_p}{\partial p(t)} \frac{\partial I(t)}{\partial p} + \frac{\partial^2 \widetilde{\sigma}_p}{\partial p(t)^2} \left(\frac{\partial p(t)}{\partial p} \right)^2 \right) d\widetilde{W}_R(t) \\ &= \left(\frac{\partial \widetilde{\mu}_p}{\partial p(t)} J(t) + \frac{\partial^2 \widetilde{\mu}_p}{\partial p(t)^2} I(t)^2 \right) dt + \left(\frac{\partial \widetilde{\sigma}_p}{\partial p(t)} J(t) + \frac{\partial^2 \widetilde{\sigma}_p}{\partial p(t)^2} I(t)^2 \right) d\widetilde{W}_R(t) \quad \text{with} \quad J(0) = 0, \end{aligned}$$

where $\frac{\partial^2 \widetilde{\mu}(t,p(t))}{\partial p(t)^2}$ and $\frac{\partial^2 \widetilde{\sigma}(t,p(t))}{\partial p(t)^2}$ are given by

$$\begin{aligned} \frac{\partial^2 \widetilde{\mu}(t, p(t))}{\partial p(t)^2} &= 6 \left(\frac{\mu_H - \mu_L}{\sigma} \right)^2 p(t) + 2 \left\{ \frac{(\mu_L - r)(\mu_H - \mu_L)}{\sigma^2} - \left(\frac{\mu_H - \mu_L}{\sigma} \right)^2 \right\}, \\ \frac{\partial^2 \widetilde{\sigma}(t, p(t))}{\partial p(t)^2} &= -2 \left(\frac{\mu_H - \mu_L}{\sigma} \right). \end{aligned}$$

Likewise, all the parameters are bounded over the interval $p(t) \in [0, 1]$. The fact $J(0) = 0$ implies that $J(t)$ can be negative.

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Figures

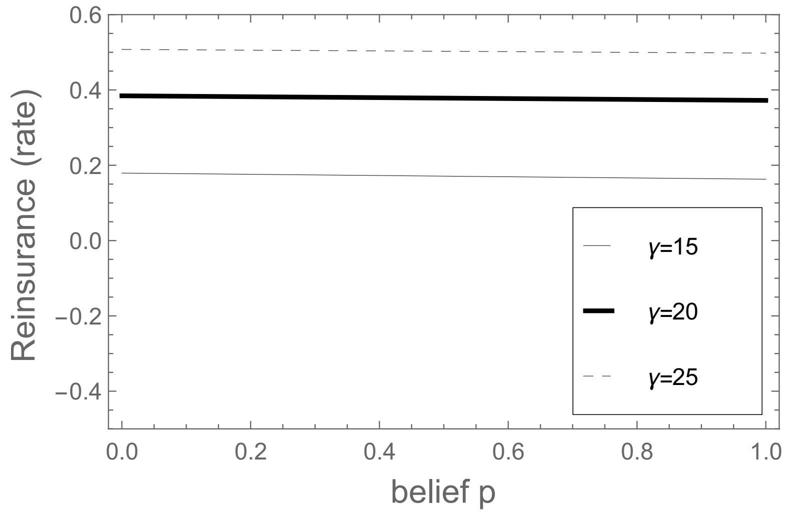
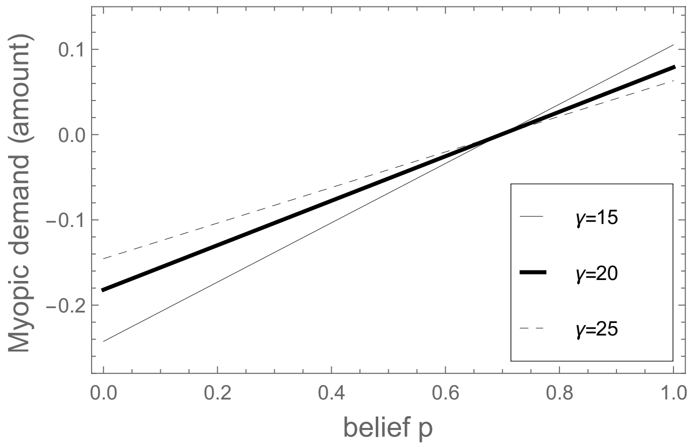
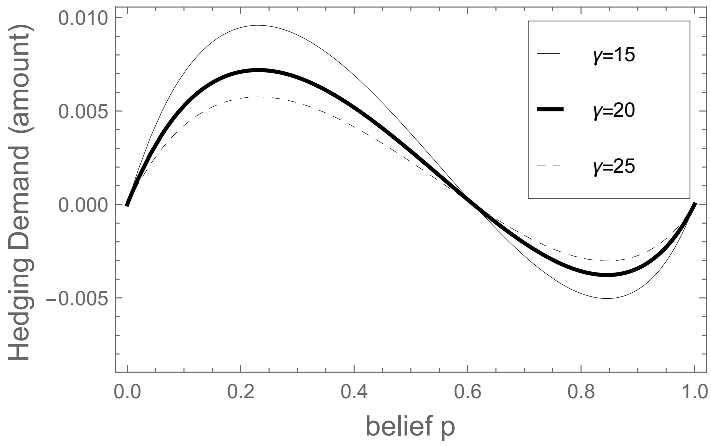


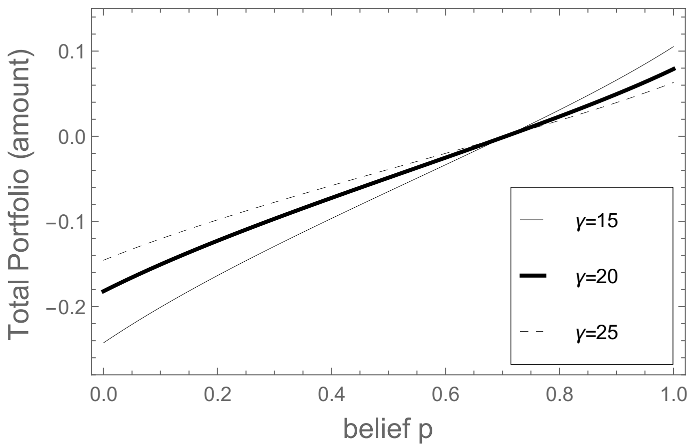
Figure 1: Optimal Reinsurance Rates ε^*. The parameters are $\mu_H = 0.1188$, $\mu_L = -0.2592$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, and $T = 5$.



(a) Myopic Demand

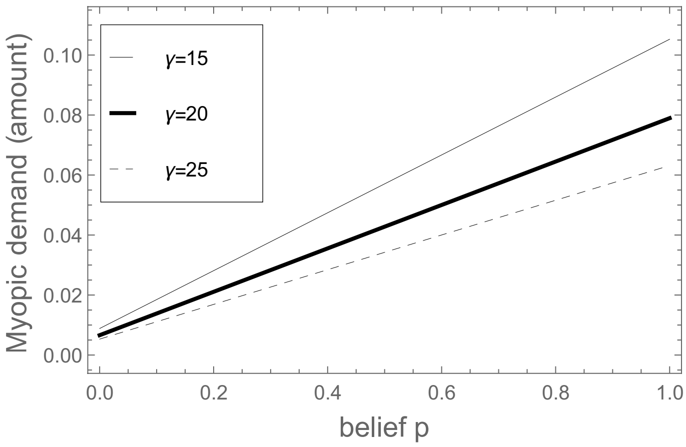


(b) Hedging Demand

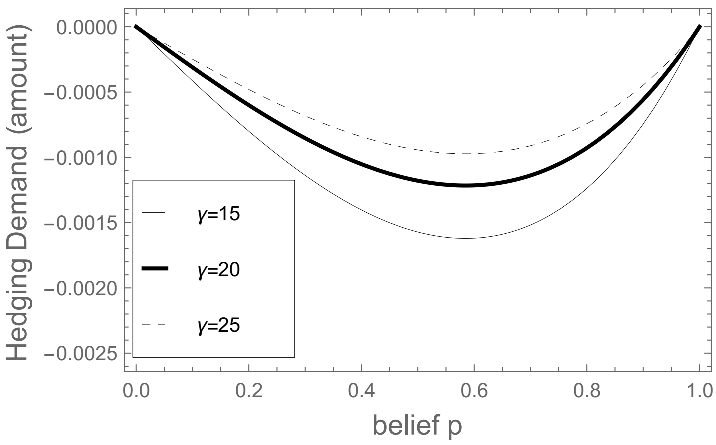


(c) Optimal Portfolio

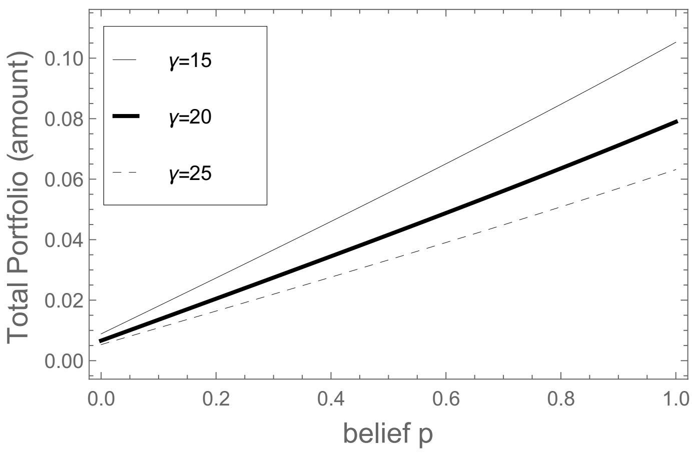
Figure 2: Optimal Portfolio Strategies (short–sale case). The parameters are $\mu_H = 0.1188$, $\mu_L = -0.2592$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, and $T = 5$.



(a) Myopic Demand

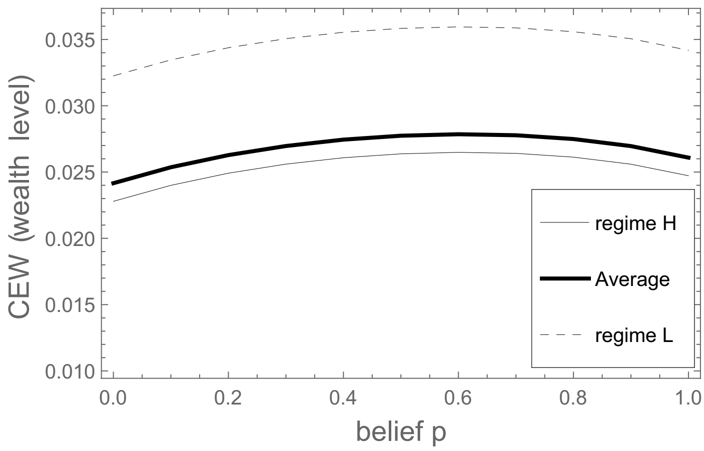


(b) Hedging Demand

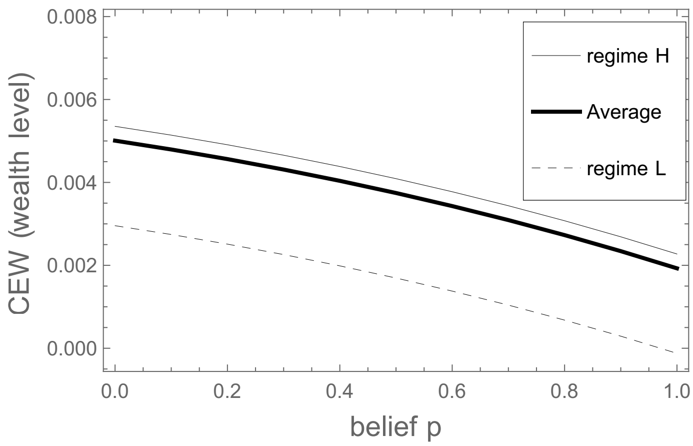


(c) Optimal Portfolio

Figure 3: Optimal Portfolio Strategies (no short–sale case). The parameters are $\mu_H = 0.1188$, $\mu_L = 0.0140$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, and $T = 5$.



(a) Short–Sale Case ($\mu_L < r$)



(b) No Short–Sale Case ($\mu_L = r$)

Figure 4: Certainty Equivalent Wealth (CEW) with $\gamma = 20$. The parameters are $\mu_H = 0.1188$, $\mu_L = 0.0140$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, $T = 5$, and $x = 0.5$.