How Should Individuals Make a Retirement Plan in the Presence of Mortality Risks and Consumption Constraints?*

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Abstract

This paper investigates optimal retirement planning when Epstein-Zin type individuals desire to maintain a certain minimum level of consumption, which can be achieved only by a guaranteed income stream after retirement. Our model incorporates the subsistence level in consumption, social securities, and defined-contribution retirement pensions, all of which are necessary to guarantee a minimal income stream. Our model shows that the movements of the optimal risky investments might dramatically change with the subsistence level in consumption. Our numerical results show that the risky investment rate in the retirement pension can increase with the risk-free gross return rate and with the risk aversion level when the low risk-free rate and risk aversion level are both low. Furthermore, the risky investment rate in the retirement pension can decrease even when the market condition is favorable.

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1. Introduction

This paper proposes an integrated portfolio management system that guarantees a certain minimal income stream. Individuals' desire for the guaranteed minimum income is incorporated as a subsistence level of consumption. The existence of this subsistence level forces our portfolio management system, or our retirement planning, to generate a steady income stream even though the system does not maximize the present market value of total income. Although many traditional retirement plans (Blanchett and Straehl, 2015[1]; Gomes and Michaelides, 2005; Horneff et al., 2008) consider annuities to provide the after-retirement income stream, they mainly focused on the market value of the future income stream rather than on its safety.[2] Different from the previous research, this paper presents an integrated retirement planning system that both maximizes the market value of the future income stream, and guarantees a minimum income stream. Our numerical result shows that the subsistence level in consumption, which is conceptually related to the minimum guaranteed income, is crucial in the optimal retirement planning because the movements of the optimal consumption and investment strategies change greatly depending on the subsistence level.

The concept of the subsistence level in consumption[3] is closely related to the minimum guaranteed income. Current consumption can be financed by an initial wealth, but all future consumption should be financed by labor income and non-labor income. This requirement means that a certain level of consumption can be maintained only by a guaranteed minimum income stream. Therefore, our model adopts the subsistence level in consumption and shows that the optimal consumption and investment decisions can dramatically change with the subsistence level in consumption.

To secure a sufficient future income stream, almost all individuals hold a social security

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[1] Our model is different from the efficient income portfolios of Blanchett and Ratner (2015) in three ways. (1) Our retirement planning is a dynamic, not a static, portfolio management, and can therefore incorporate the individuals' elasticity of intertemporal substitution in consumption. (2) Our problem is mainly long-term portfolio management for a comfortable life after retirement, not an efficient short- or intermediate-term horizon investment. (3) The stability of income in our model is mainly related to income variation across the time, not across the state.

[2] Dai et al. (2008) and Milevsky and Salisbury (2006) study the financial value of guaranteed minimum withdrawal benefits.

[3] To explain the equity premium puzzle, Constantinides (1990) also incorporates the subsistence level in consumption.

pension[4] and a (defined-contribution) retirement pension[5]. Social security, which is mandatory for all individuals, requires a fixed amount of saving until retirement and promises fixed regular payments until death. In contrast, the defined-contribution (DC) retirement pension's payments can change depending on their investment performance whereas the retirement pensions require a fixed amount of saving. Although social security and the retirement pension have these characteristics, traditional optimal lifetime consumption and investment problems have excessively simplified their structures (Chen et al., 2006; Huang and Milevsky, 2008; Milevsky and Young, 2007). However, our model integrates all of these characteristics of social securities and retirement. (Figure 1)

[Insert Figure 1 here.]

Intuitively, saving in the social security and the retirement pension has a different purpose from the direct asset allocation into equities.[6] Individuals hold the social security for a minimum guaranteed income. Governments provide social security for the welfare of their retirees; it guarantees a minimal, but not sufficient, amount of income stream. Saving in the DC-type retirement pension might corresponds to individuals' desire for the conservatively flexible income. The retirement pension is both conservative in that it makes a fixed and regular payments like a coupon bond, and flexible in that individuals can choose their investment strategy in the pension account. On the other hand, the traditional asset allocation between risk-free and risky assets can be considered as an investment for a desired additional income. After preparing a minimum after-retirement life by saving into such pensions, individuals can invest a portion of their surplus wealth into risky assets. Therefore, in constructing an after-retirement income stream, both the social security and the retirement pension, as well as a traditional investment, should be considered.

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[4] United Kingdom offers State Basic Pension and State Second Pension as a social security. As a social security of United States, Old-Age, Survivors and Disability Insurance Program (OASDI) is offered.

[5] According to Bureau of Labor Statistics (July, 2013) report," Employee Benefits in the United States", most full-time American workers and employees of large companies participate in DC-type plans.

[6] This interpretation is based on the philosophy of Merton (2003, 2014). Although Merton (2003, 2014) mainly dealt with the inflation risk, Merton (2003, 2014) and we share one very important thing in common, intertemporal variation, as well as risk, of income stream is significant in the construction of the life-time optimal consumption and investment strategies.

[Insert Figure 2 here.]

Representative individuals of this paper are characterized by Epstein-Zin (1989) type utility. Their whole lifetime is divided into 4 periods, each time node is indexed by numbers from 0 to 4. (Figure 2) We assume that the individuals are exposed to the mortality risk and that they surely retire at time 2. Therefore, the individuals choose an optimal investment strategy in the DC-type retirement pension at time 0 and 1 (before they retire) and, for whole lifetime, they construct optimal consumption and investment strategies if they are alive. We mainly investigate these optimal strategies' sensitivity to the investment environment and the individuals' preference to risk and intertemporal variance.

Our numerical results show that the risky investment in the retirement pension has two opposite effects on the direct investment in the risky asset: a complementary effect and a substitution effect. The complementary effect is that investment in the retirement pension reinforces the total investment amount in the risky asset. When the complementary effect is dominant, individuals raise both the risky investment rate in the retirement pension and the direct risky investment rate, so the total risky investment amount grows dramatically. In this case, we can say that individuals use the retirement pension for a desired additional income. In contrasts, the substitution effect decreases the amount of direct investment in the risky asset. When the substitution effect dominates the complementary effect, the total risky investment amount decreases even though the risky investment in the retirement pension increases. This relationship occurs mainly because risk-averse individuals prefer the risky investment in the retirement pension, which can partially hedge the mortality risk of individuals, to the direct risky investment. To get a steady and sufficient annuity income stream from the retirement pension rather than a high return on the total wealth, individuals will increase the risky investment in the retirement pension and decrease the direct risky investment. In this case, we can say that individuals use the retirement pension for a guaranteed minimum income.

These analyses suggest that the retirement pension can be used for different objectives (i.e., for a guaranteed minimum income or for a desired additional income), depending on the investment environment and the individuals' risk preferences. The risky investment in the retirement pension is fundamentally a part of the investment in the risky asset, so the retirement pension is usually used as a desired additional income. However, we observe that individuals save a part of their wage as retirement pension to obtain a guaranteed minimum income when the risk-free gross return is low and when the individuals are less risk-averse. In both cases, the risky investment in the retirement pension increases with the risk-free gross return rate and with the risk aversion level, whereas the total risky investment amount decreases trivially. This observation implies that the guaranteed sufficient annuity income is more important than the high expected return of the total wealth, and that the risky investment in the retirement pension is related to the guaranteed minimum income.

Furthermore, our numerical result shows that the individuals can reduce the risky investment in the retirement pension when the probability with high return of the risky asset is excessively high. Because the future income stream from the retirement pension is exposed to mortality risk as well as to market risk, the individuals might prefer the direct risky investment to the risky investment *via* the retirement pension (an inverse substitution effect). From this observation, we can say that the traditional asset allocation, which includes the direct risky investment, is more closely related to the desired additional income than is the retirement pension. Following Merton's categories, we assert that the retirement pension is related to the conservatively flexible income.

2. Model

2.1. Financial Markets and Wealth

We employ a 4-period ($n = 0,1,2,3,4$) discrete-time binomial tree model and assume that the financial market has a risk-free asset with a constant gross return rate R and a risky asset with a stochastic return rate: the time-n gross return rate α_n ($n = 1,2,3,4$) of the risky asset is α^u with probability p^u and α^d with probability $p^d = 1 - p^u$. (u and d represent up and down markets, respectively.) The gross return rates of the risk-free and the risky assets are assumed to satisfy the so-called *no-arbitrage condition* of $\alpha^u > R > \alpha^d$.

We denote time-n wealth level of individuals as W_n ($n = 0,1,2,3,4$). The individuals in our model is assumed to have initial wealth $W_0 = W$, and receives wage incomes I_0 and I_1 at times $n = 0$ and $n = 1$, respectively. They are also assumed to retire at time $n = 2$ and thus does not receive any wage incomes after that time. Before retirement, they are forced to save a portion of their wage income with given rates θ_n^S in the social security and θ_n^{RP} into the retirement pension.[7] We consider a DC retirement pension, not a defined-benefit (DB)

[7] Although, in reality, individuals can choose the amount of saving in the retirement pension, such as 401(k), we

retirement pension,[8] the individual has a right to choose the risky investment rate π_n^{RP} $(n = 0,1)$ in the retirement pension. After saving in the social security and retirement pension and making their asset allocation in the retirement pension, the individuals sequentially choose the consumption level C_n and the risky investment rate π_n of the surplus wealth $\left(W_n - C_n + (1 - \theta_n^S - \theta_n^{RP})I_n\right)$. When we denote time-$n$ wealth level in the retirement pension by W_n^{RP} with initial wealth $W_0^{RP} = 0$, the relationships between the two subsequent wealth levels, (W_{n-1}, W_{n-1}^{RP}) and (W_n, W_n^{RP}), before the retirement are given as follows:

$$\begin{cases} W_n = (\alpha_n - R)\pi_{n-1}\left(W_{n-1} - C_{n-1} + \left(1 - \theta_{n-1}^S - \theta_{n-1}^{RP}\right)I_{n-1}\right) + R\left(W_{n-1} - C_{n-1} + \left(1 - \theta_{n-1}^S - \theta_{n-1}^{RP}\right)I_{n-1}\right), \\ W_n^{RP} = (\alpha_n - R)\pi_{n-1}^{RP}\left(W_{n-1}^{RP} + \theta_{n-1}^{RP}I_{n-1}\right) + R\left(W_{n-1}^{RP} + \theta_{n-1}^{RP}I_{n-1}\right), \end{cases}$$

for $n = 1,2$.

After retirement, the individual's wage income is replaced by annuity income from the social security and the retirement pension. The social security income is assumed to be proportional to the last wage income I_1 with a given constant rate ρ^S. In contrast, the regular payment I^{RP} of the retirement pension is fairly determined; i.e., the expected value of discounted sum of incomes from the retirement pension

$$I^{RP} + (1 - \delta'_3)I^{RP}/R$$

should be equal to the wealth level W_2^{RP} at retirement, $n = 2$:

$$W_2^{RP} = I^{RP} + (1 - \delta'_3)I^{RP}/R,$$

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suppose that the savings-to-income ratio θ_n^{RP} is exogenously given. This is because we want to focus on the risky investment ratio in the retirement pension. Instead, we adopt the recommended saving ratio of their wage income into 401k, which is the most popular retirement pension.

[8] Milevsky and Young (2007) consider only a DB retirement pension. However, we choose a DC type because, after dot-com crash in 2000, the shift from DB pensions to DC pensions has accelerated. In 2011 (In 1979), 31% (7%) of all private-sector workers participated only in a DC and 3% (28%) participated only in a defined benefit (DB) pension plan. (11% (10%) percent had both a DC and a DB plan, and the residual percentage is the fraction of private-sector wage and salary workers who were NOT a participant in an employment-based retirement plan)

so

$$\rho^{RP} = \frac{1}{1+(1-\delta'_3)/R'}$$

where δ'_n $(n = 1,2,3,4)$ is the time-n *risk-neutral* probability of individual's death. Now, we can construct the relationships between two subsequent wealth levels W_{n-1} and W_n, after retirement with these annuity incomes $\rho^S I_1$ and $\rho^{RP} W_2^{RP}$:

$$W_n = (\alpha_n - R)\pi_{n-1}(W_{n-1} - C_{n-1} + \rho^S I_1 + \rho^{RP} W_2^{RP}) \\ + R(W_{n-1} - C_{n-1} + \rho^S I_1 + \rho^{RP} W_2^{RP}), \qquad \text{for } n = 3,4.$$

2.2. A Lifetime Consumption and Investment Problem

Based on the wealth process, the individuals choose consumption and investment strategies that maximize their utility, and the utility preference is assume to be an Epstein-Zin (1989) type. Therefore, optimal continuation value function of the individual is written as, for the time-n *real* probability of individual's death, δ_n $(n = 1,2,3,4)$,

$$V_n^L = \max_{C_n \geq \underline{C}_b, \pi_n, \underline{\pi} \leq \pi_n^{RP} \leq \overline{\pi}} \left(c_n^{1-\rho} + \beta \left(\delta_{n+1} \mathbf{E}_n \left[{V_{n+1}^D}^{1-\gamma} \right] + (1-\delta_{n+1}) \mathbf{E}_n \left[{V_{n+1}^L}^{1-\gamma} \right] \right)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}, \quad (1)$$

for $n = 0,1$,

$$V_n^L = \max_{C_n \geq \underline{C}_a, \pi_n} \left(c_n^{1-\rho} + \beta \left(\delta_{n+1} \mathbf{E}_n \left[{V_{n+1}^D}^{1-\gamma} \right] + (1-\delta_{n+1}) \mathbf{E}_n \left[{V_{n+1}^L}^{1-\gamma} \right] \right)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}, \quad (2)$$

for $n = 2,3$, and

$$V_n^D = \begin{cases} W_n + W_n^{RP} + \sum_{k=0}^{n-1} \theta_k^S I_k, & \text{for } n = 1,2, \\ W_n, & \text{for } n = 3,4, \end{cases} \quad (3)$$

subject to

$$\begin{cases} W_n = (\alpha_n - R)\pi_{n-1}\left(W_{n-1} - C_{n-1} + \left(1 - \theta_{n-1}^S - \theta_{n-1}^{RP}\right) I_{n-1}\right) + R\left(W_{n-1} - C_{n-1} + \left(1 - \theta_{n-1}^S - \theta_{n-1}^{RP}\right) I_{n-1}\right), \\ W_n^{\mathrm{RP}} = (\alpha_n - R)\pi_{n-1}^{\mathrm{RP}}\left(W_{n-1}^{\mathrm{RP}} + \theta_{n-1}^{RP} I_{n-1}\right) + R\left(W_{n-1}^{\mathrm{RP}} + \theta_{n-1}^{RP} I_{n-1}\right), \end{cases}$$

for $n = 1,2$, and

$$W_n = (\alpha_n - R)\pi_{n-1}(W_{n-1} - C_{n-1} + \rho^S I_1 + \rho^{RP} W_2^{RP}) + R(W_{n-1} - C_{n-1} + \rho^S I_1 + \rho^{RP} W_2^{RP}),$$

for $n = 3,4$,

$$C_n \geq \begin{cases} \underline{C}^b & \text{for } n = 0,1, \\ \underline{C}^a & \text{for } n = 2,3, \end{cases} \tag{4}$$

and

$$\underline{\pi} \leq \pi_n^{RP} \leq \overline{\pi} \qquad \text{for } n = 0,1, \tag{5}$$

where β $(0 < \beta < 1)$ represents the subjective discount rate, γ $(\gamma > 0, \gamma \neq 1)$ is the relative risk aversion (RRA) level, and $\eta = 1/\rho$ $(\eta > 0,\ \eta \neq 1)$ means the level of elasticity of intertemporal substitution (EIS) in consumption. In equations (1) and (2) the continuation value function, V_n^L, is defined as the maximized equivalent wealth level of the individual. Following its definition we define the value function V_n^D at death as a wealth level at that moment as described in equation (3), and thus, it naturally reflects the bequest motive of the individual.

The consumption constraints of (4) stand for the requirement of a minimum consumption level to sustain life, and the two subsistence levels, $\underline{C}^b$ and $\underline{C}^a$, before and after retirement restrict the possibilities of substituting consumption intertemporally. The portfolio constraints of (5) are usually imposed in their law for fund safety. Anyone cannot go short in a US DC plan, so the equity allocation would range between 0%-100%.

3. Data and Parameter Estimation

We display baseline parameters in Table 1.

[Insert Table 1 here.]

Because the time interval $\Delta t = 15$ years of our problem is rather large, we carefully chose the parameters R, α^u, α^d, and p^u related to financial market conditions.

The 15-year risk-free gross return $R = 1.33 = 1.0192^{15}$ is based on Ibbotson's Capital Market Assumptions (CMAs) as of December 31, 2011. The assumptions reported that the annual expected rate return rate of cash (IA SBBI US 30-Day TBill TR USD) is 1.92%.

The stock parameters α^u, α^d, and p^u are estimated by choosing the values that minimize the sum of squared errors of the mean and the standard deviation:

$$\min_{\alpha^u,\alpha^d,p^u} \left\{ \left(\text{Mean} - (p^u\alpha^u + (1 - p^u)\alpha^d) \right)^2 + \left(\text{Std.} - \sqrt{(p^u(\alpha^u)^2 + (1 - p^u)(\alpha^d)^2) - (p^u\alpha^u + (1 - p^u)\alpha^d)^2} \right)^2 \right\},$$

subject to $\alpha^u > R > \alpha^d > 0$ and $0 < p < 1$. Here, Mean and Std. are the mean and standard deviation of the gross return of the S&P 500 Index. To calculate them, we used the closing price of the S&P 500 Index on a yearly basis from *Yahoo Finance*. We generated two 15-year S&P 500 Index gross return data sets: the first data set contains 15-year gross return over a moving window; the second contains 15-year gross return from non-overlapping periods. Estimation results using the two data sets were quite similar and both sums of squared errors are significantly negligible. (See Table 2) We employ the parameters in the moving-window case as a baseline: $\alpha^u = 4.28534$, $\alpha^d = 0.409691$, and $p^u = 0.722719$.

[Insert Table 2 here.]

The individual's initial wealth level W is estimated based on the data of the 2013 Survey of Consumer Finances (SCF). We are interested in the optimal lifetime consumption and investment strategies of individuals aged 35 years, and thus, use the estimates[9] of the group aged 35 to 44; i.e., $W = \$347200$. The wage income level I is obtained from the data of Labor Force Statistics from the Current Population Survey.[10] We use the median weekly earnings of full-time wage and salary workers to estimate the wage income level. We calculate 15-year wage levels $I_0 = \$630308$ and $I_1 = \$644511$ by adding all discounted annual wage for 15 years:

[9] We use the mean value of net worth for families with holdings as estimates for the wealth levels.

[10] One can access to this statistics via Bureau of Labor Statistics.

$$I_n = \sum_{k=0}^{14} \frac{I_{n,k}^a}{R_a^k} = \frac{I_{n,k}^a \left(1 - \frac{1}{R_a^{15}}\right)}{1 - \frac{1}{R_a}}, \qquad \text{for } n = 1,2,$$

where $R_a = 1.01192\%$ is the gross annual risk-free return rate and $I_{n,k}^a$ is the annual wage income level at age $(20 + 15n + k)$.

We set the parameters related to the social security and the retirement pension as $\theta_1^S = 7.19\%$ and $\theta_2^S = 10.94\%$. The amount saved in the social security is mandated and we just assume that the saving rate starts at 3% at age 25 and geometrically increases by 0.25% per year until retirement. As we convert the annual wage level to the 15-year wage level, we also adjust the annual saving rate for the social security to the time interval $\Delta t = 15$ of our model. The payment-to-income ratio ρ^S of the social security is assumed to be 43%, which is calculated on the homepage of American Association of Retired Persons (AARP). We assume that the individual saves 10% of their wage income into the retirement pension throughout their career; these rates are commonly recommended in 401k, which is the most popular retirement pension. Because 401k has no position limits, we just assume $\bar{\pi} = 1.0$ and $\underline{\pi} = 0.0$, excluding leverages and short positions. The mortality rate, which is essential to the calculation on the retirement pension payment, is obtained from the Social Security Periodic Life Table, which is publicly available: $\delta_1 = 0.04, \delta_2 = 0.13, \delta_3 = 0.381$, and $\delta_4 = 1.0$.

The annual subjective discount factor is assumed to be 0.99. Therefore, the 15-year subjective discount factor $\beta = 0.86 = 0.99^{15}$. The coefficients of the RRA and EIS in consumption are assumed to be 5 and 1/3, respectively.[11]

4. Implication

Using the baseline parameters in Table 1, we calculated the individual's optimal lifetime consumption and investment strategies of the consumption-to-wealth ratio C^*/W, the risky investment rate π^*, and the risky investment rate π^{RP^*} in the retirement pension, as functions

[11] According to the results of Vissing-Jorgensen (2002), the value of EIS is around 0.3-0.4 for stockholders.

of the subsistence level $\underline{C}^a/W = \underline{C}^b/W$ for $0 \leq C^*/W \leq 2$ (Figure 3). The subsistence level in consumption is defined as a mode of consumption that corresponds to the basic needs of life. The basic needs considered in this paper includes welfare as well as a dietary needs, so the interpretation of our subsistence level corresponds to the 'weak' poverty line, not the 'strong' poverty line which is used to identify that part of population that is regarded as absolutely poor.[12] The median income-to-wealth ratio of the group with income from the bottom 20% to the bottom 40% is 1.13 and the median ratio of the group with income from the bottom 40% to the bottom 60% is 1.85. Neither group is absolutely poor, but can be considered as 'weakly' poor in some sense. Therefore, we guess that an appropriate subsistence level in C^*/W might be between 1 and 2.

[Insert Figure 3 here.]

Figure 3 plots the optimal consumption and investment strategies for a wide range of subsistence level from 0 to 2, and we can observe that they are trivial when the subsistence level in C^*/W is quite small (especially, less than 1) or large (larger than about 1.6). Therefore, we choose two subsistence levels, a low (1.2) one and a high (1.5) one, to further investigate the effect of the subsistence level on the optimal consumption and investment behaviors. Along the subsistence levels in consumption-to-wealth ratio, the optimal consumption and investment patterns as functions of investment opportunities and individual's preferences change greatly. This trend means that optimal consumption and investment decisions should be adjusted depending on the individuals' subjective subsistence levels.

Case 1: Low subsistence level 1.2 in consumption-to-wealth ratio

Optimal consumption and investment strategies for this case vary with risk-free gross return R (Figure 4). As has been seen in traditional optimal investment problems, the optimal risky investment rate π^* decreases with the risk-free gross return R. This relationship is natural because, as the risk-free gross return increases, the risky asset becomes decreasingly attractive to individuals. However, the optimal risky investment rate π^{RP^*} in the retirement pension increases with the risk-free gross return R when the risk-free gross return is considerably low.

[12] Steger (2000) interprets the subsistence level as the strong poverty line.

Based on this numerical result, we can say that the retirement pension also contributes to generation of a minimum guaranteed income stream that is necessary for subsistence level in consumption. When the risk-free gross return is not high enough, the present value of the individuals' future income decreases, so they think that their wealth is too low to sustain subsistence level in consumption after their retirement. When the risk-free gross return is low, some future optimal consumption-to-wealth ratios C_n^*/W at a future down state bind to the subsistence level $\underline{C}^{\mathrm{a}}/\mathrm{W} = \underline{C}^{b}/W$ in the consumption-to-wealth ratio. In this case, the individuals will prefer to increase the proportion of their wealth that is invested in the risky asset for a high-risk-high-return investment. With increase in the riskiness of investment in the retirement pension, which is expected to maximize the wealth W_2^{RP} in the retirement pension, individuals expect to maintain the optimal consumption amount above the subsistence level for any economic state.

In contrast, when the risk-free gross return R is high enough, the retirement pension seems to generate a desired additional income stream. Without using the retirement pension, individuals can match the minimum guaranteed income stream by allocating assets directly to the risk-free asset and the risky asset. In this circumstance, individuals choose the optimal risky investment rate in the retirement pension to maximize the risk-adjusted total return rate without any constraints. When the risk-free gross return is high, individuals can obtain the maximum risk-adjusted return even with small exposure to risk, so the investment rate in the retirement pension decreases with the risk-free gross return R.

[Insert Figure 4 here.]

In case with a low subsistence level, optimal consumption and investment strategies is not much different from those of the traditional ones: the investment and consumption ratios increase with the up probability p^u (Figure 5, left and right) and decrease with the RRA level γ (Figure 6, left and right). Because the investment environment becomes increasingly positive as p^u increases, the two optimal investment rates $\pi^*$and π^{RP^*} increase monotonically with p^u. In contrast, optimal investment rates π^* and π^{RP^*} decrease with γ because the attractiveness of the risky asset decreases as the risk-aversion of the individuals increases. Since the expected rate of return of the investment decreases with the decrease in the investment rates π^* and π^{RP^*}, optimal consumption-to-wealth ratio C^*/W also decreases.

The graphs of the optimal risky investment rate π^* are steep when the graphs of the optimal risky investment rate π^{RP^*} in the retirement pension are flat. This relationship implies that both the direct risky investment and the risky investment in the retirement pension increase the total risky investment. When the risky investment rate π^{RP^*} binds to any boundaries $\overline{\pi}$ or $\underline{\pi}$, the risky investment rate π^{RP^*} cannot change any more, so instead the direct risky investment rate π^* changes more dramatically.

[Insert Figure 5 here.]

[Insert Figure 6 here.]

As the RRA level does, the EIS level also has a non-linear effect on both investment rates π^* and π^{RP^*}. (Figure 7, left and middle[13]) Individuals with low EIS levels are more averse to intertemporal variation in consumption than those with high EIS levels. Therefore, individuals with a low EIS level prefer the risky investment in the retirement pension to the direct risky investment. This is because both investments make individuals exposed to the 'same' risky asset, but only the retirement pension generates a steady income stream after individuals' retirement. In this sense, the risky investment rates π^* and π^{RP^*}, respectively, increases and decreases when the level of EIS is low. In other words, the risky investment in the retirement pension substitutes the direct risky investment when individuals are more averse to intertemporal variation, i.e., when the EIS level is low. However, this tendency is reversed with high EIS levels. Individuals with a high EIS level does not severely averse to intertemporal variation. The only matter which they consider is risk, not intertemporal variation. Therefore, they increase the saving in the risk-free asset, so the risky investment ratio π^* decreases with the EIS level. As a substitute, the risky investment ratio π^{RP^*} increases with the level of EIS.

[Insert Figure 7 here.]

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[13] We plot the graphs only with EIS levels between zero and one. When the EIS level is larger than unity, the optimal strategies π^*, π^{RP^*}, and C^*/W are trivial, so we do not present the numerical results.

Case 2: High subsistence level 1.5 in consumption-to-wealth ratio

When the subsistence level in consumption-to-wealth ratio was high, the plots of optimal strategies π^*, π^{RP^*}, and C^*/W functions of p^u (Figure 8) and γ (Figures 9) showed interesting responses. In the graph of risky investment rate π^{RP^*} in the retirement pension as a function of p^u (Figure 8, middle), the risky investment rate π^{RP^*} in the retirement pension increases until $p^u \approx 0.7$, then decreases. The increase in this graph can be explained as a consequence of the improvement of investment opportunity. Especially, when the investment opportunity is bad, the present value of their future annuity income decreases, so to support future subsistence level in consumption, they increase their risky investments as p^u increases. This explanation is confirmed by the consumption-to-wealth ratio graph in Figure 8. During the risky investment π^{RP^*} increases, C^*/W binds to the subsistence level.

After p^u becomes high enough, the risky investment rate π^{RP^*} in the retirement pension does not increase, but decreases even though the market condition improves. This response is a consequence of the mortality risk. The total earning from the retirement pension changes depending on individual's death time, so we can say the exposed amount of mortality risk is proportional wealth level W_2^{RP} at time 2 in the retirement pension. As p^u increases, individuals overvalue the wealth in the retirement pension. This behavior leads to the increase in the volatility of the earnings from the retirement pension because the amount of the earnings varies due to the mortality risk, therefore risk-averse individuals prefer the risk-free asset in the retirement pension.

[Insert Figure 8 here.]

The optimal investment rate π^{RP^*} in the retirement pension as a function of RRA, γ, is also hump-shaped (Figure 9, middle). The optimal investment rate π^{RP^*} in the retirement pension increases even when γ increases. Although this observation seems unnatural because the investment rates usually decrease with γ, this trend can be explained by the influence of risky investment in the retirement pension as a substitution for the direct investment in a risky asset.[14] Total investment in the risky asset decreases as individuals' risk-aversion increases.

[14] Recall that in our model individuals can invest in a risky asset in two ways: direct investment and investment

However, individuals can increase the investment in the retirement pension by dramatically decreasing the direct investment. Because the structural characteristics of the retirement pension can partially hedge the longevity risk, the individual can reduce the total risk by increasing their investment in the retirement pension, instead of by directly investing in the risky asset. The striking decrease in the investment rate π^* with a low γ (Figure 9, left) is consistent with this explanation.

[Insert Figure 9 here.]

[Insert Figure 10 here.]

High subsistence level alleviates the effect of individuals' desire for consumption smoothing, (or equivalently, the EIS effect) on the optimal investment rate π^*. (Figure 10, left) With a high subsistence level, the individuals' current and future consumptions more likely bind to the subsistence level, so the optimal consumption is automatically smoothed. Therefore, the optimal investment rate π^* (Figure 10, left) becomes more insensitive to the level of EIS, compared to the optimal investment rate π^* in the case with a low subsistence level (Figure 7, left). On the other hand, the increase in the optimal investment rate π^{RP^*} in the retirement pension (Figure 10, middle) can be explained in a traditional manner. Increasing in the EIS level means that the individuals becomes less averse to intertemporal variation. Then, they will prefer a more risky investment in the retirement pension, which is expected to support higher levels of consumption after their retirement.

[Insert Figure 11 here.]

The optimal strategies π^*, π^{RP^*}, and C^*/W as functions of risk-free gross return R differ according to the subsistence level (Figures 3 and 11). The major difference is observed in optimal risky investment rate π^* (Figures 3 and 11, left). With subsistence level 1.5 (Figure 11), the optimal risky investment rate π^* does not decrease monotonically. When the risk-free

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through the retirement pension.

gross return is considerably low, individuals increase both the direct risky investment amount and the risky investment amount in the retirement pension. This response occurs because as the risk-free rate increases, the minimal wealth in the risk-free asset required for future minimum consumption level decreases. Consistently, the optimal consumption-to-wealth ratio C^*/W stays in the subsistence level while the optimal risky investment rate π^* increases. After the risk-free gross return gets high enough, the individual currently consumes more than the subsistence level because the future subsistence levels can be supported by the surplus wealth after the current consumption.

Summary of Numerical Results

In sum, the consideration of a moderately high subsistence level in a life-cycle model can lead to optimal consumption and investment strategies for individuals who want to improve the stability of future income stream. When subsistence level is low ($\underline{C}^a/W = \underline{C}^b/W = 1.2$), the risky investment in the retirement pension only increases with a low risk-free gross return, but when subsistence level is high ($\underline{C}^a/W = \underline{C}^b/W = 1.5$), both the direct risky investment and the risky investment in the retirement pension monotonically decrease. The difference in trends occurs because individuals' motive to achieve a minimum guaranteed (regardless to a future market state) income intensifies as the subsistence level in consumption increases. Therefore, we can say that when subsistence level is high, the top priority of individuals' financial management is to guarantee a minimum income stream.

The same effect of the high subsistence level on the optimal investment strategy is observed again in other figures. When the risk-free gross return rate is considerably low, the direct risky investment decreases (increases) with the risk-free gross return rate R when subsistence level is low (high). (Figures 3 and 11, left) In case with high subsistence level, individuals use the risk-free asset for guaranteeing a future minimal income level, not for making additional risk-free financial income. Therefore, individuals reduce the necessary saving amount in the risk-free asset, which is required to match the minimal guaranteed income, as the risk-free gross return rate increases when their subsistence level is high. This is an opposite result in case of low subsistence level, which is consistent to the result of traditional Merton's (1971) result.

When it comes to the optimal investment rate π^{RP^*} in the retirement pension, the effect of the subsistence level is observed only when the subsistence level was high (Figures 8 and 9, middle); when it is low, the optimal investment strategy in the retirement pension (Figures 5 and 6, middle) were definitely similar to that of classical Merton's (1971) problem, that does not consider the subsistence level. The risky investment rate π^{RP^*} in the retirement pension very noticeably increases along with the increase in γ only when the subsistence level is sufficiently high (Figure 9, middle). Because the risky investment rate π^* decreases with γ as usual, we can conclude that in this case the risky investment in the retirement pension has different purpose from the direct risky investment. When the subsistence level is considerably high, individuals directly invest in the risky asset to generate the required additional income, but they allocate their wealth in the retirement pension into the risky asset to stabilize the after-retirement income stream, which is less risky than the return from the direct risky investment.

This result indicates that when individuals hold a retirement pension for the purpose of guaranteeing an income, the risky investment in the retirement pension can *substitute* the direct risky investment for own purpose. Both risky investments increase the risk to which the individual is currently exposed, but give different payoffs after the individuals' retirement. When the individuals have a strong motive to stabilize their future income stream, they prefer the risky investment in the retirement pension to the direct risky investment. As a consequence, they increase the risky investment rate π^{RP^*} in the retirement pension and decrease the investment rate π^*; i.e., the risky investment in the retirement pension has a *substitution effect* on the risky investment.

The substitution effect is not the only consequence of risky investment in the retirement pension. Basically, the risky investment in the retirement pension has a *complementary effect* on the direct investment in the risky asset; i.e., the investment in the retirement pension reinforces the total investment amount in the risky asset. This effect occurs because the risky investment in the retirement pension is a part of total risky investment. As the risky investment rate π^{RP^*} in the retirement pension increases, the total investment rate also increases as long as the change of the investment in the retirement pension does not offset the change in the direct investment. This complementary effect is clearly observed when the subsistence level was low. (Figures 5 and 6)

The complementary and the substitution effects are, respectively, related to the two different purposes of the retirement pension: the additional-income purpose and the guaranteed-income purpose. When the individuals require additional income from the retirement pension as they expect in the direct risky investment, the complementary effect is dominant. On the contrary, when the individuals want to make a minimum-guaranteed income from the retirement pension, the substitution effect becomes dominant; i.e., the prominence of the substitution effect increases when the individuals have a high subsistence level in consumption. Finally, we can say that retirement planning that does not consider the subsistence level in consumption can lead to an inappropriate investment strategy in the retirement pension when the individuals mainly want to receive a guaranteed stable income stream after retirement.

5. Conclusion

Our model proposes an integrated retirement plan that maximizes the market value of a future income stream and that guarantees a minimum income stream. Our main contribution is to show that the incorporating subsistence level in consumption is essential for a proper lifetime consumption and investment strategies which guarantee a minimum income stream. In other words, the optimal lifetime consumption and investment strategies of an individual are influenced by the guaranteed minimum income stream that the individual desires.

Our numerical results show that, depending on the market environment, individuals hold a retirement pension for different purposes: either to guarantee a minimum income or to provide additional desired income. The amount of the annuity from the retirement pension does not depend on the economic states after individuals' retirement, so when the financial market is depressed, individuals use the retirement pension to prepare a stable after-retirement income stream. In contrast, when the market condition is favorable, the retirement pension is used to generate a desired additional income because individuals can support the subsistence level in consumption only by investing in a risk-free asset and a risky asset.

Finally, our model confirms that the subsistence level in consumption must be considered when developing an optimal retirement plan. The optimal behaviors that our model predicts for reasonable subsistence levels in consumption are different from the predictions of the classical optimal consumption and investment behaviors. Our numerical results demonstrate that the risky investment rate in the retirement pension can increase even when the low risk-free rate or the low risk aversion level increase and that the risky investment rate in the retirement pension can decrease even in a prosperous market condition.

Appendix A. The Sketch of the Algorithm to Solve the Problem

Equations (1) and (2) define the value function of our problem recursively. Therefore, in those equations, time-n continuation value function V_n^L depends on the optimal strategies π_n^*, $\pi_n^{\mathrm{RP}^*}$, C_n^*/W, the time-$(n+1)$ value function V_{n+1}^D at death, and the continuation value function V_{n+1}^L. Here, time-$(n+1)$ continuation value function V_{n+1}^L depends on the next-time optimal strategies π_{n+1}^*, ${\pi_{n+1}^{\mathrm{RP}}}^*$, and C_{n+1}^*/W. Therefore, time-n continuation value function V_n^L depends on time-n or time-$(n+1)$ optimal strategies π_n^*, $\pi_n^{\mathrm{RP}^*}$, C_n^*/W, π_{n+1}^*, ${\pi_{n+1}^{\mathrm{RP}}}^*$, and C_{n+1}^*/W. Repeating this logic, we can represent the current value function V_0^L with a maximization operator with respect to all choice variables at any node in our binomial tree model. Solving this maximization problem combined with (3), (4), and (5), yields the solution of our model. All results in this paper were calculated using the optimization toolbox and the global optimization toolbox of Matlab.

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Figures

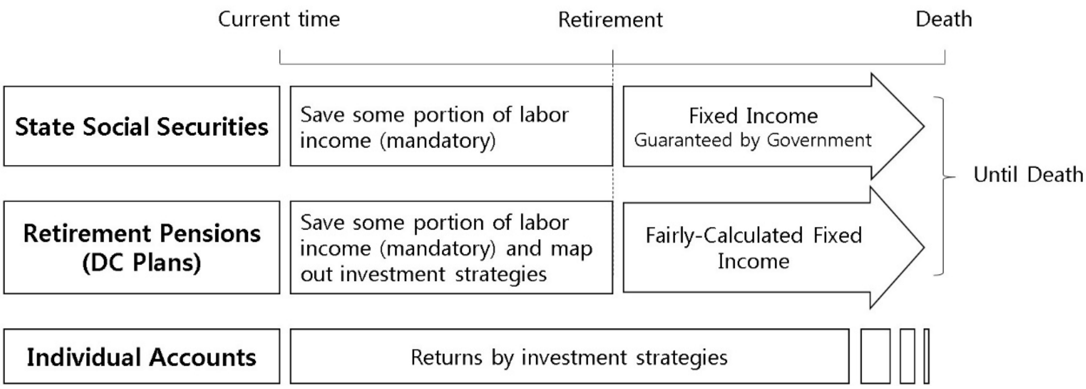

Figure 1. Structural Characteristics of Social Securities and DC-type Retirement Pensions

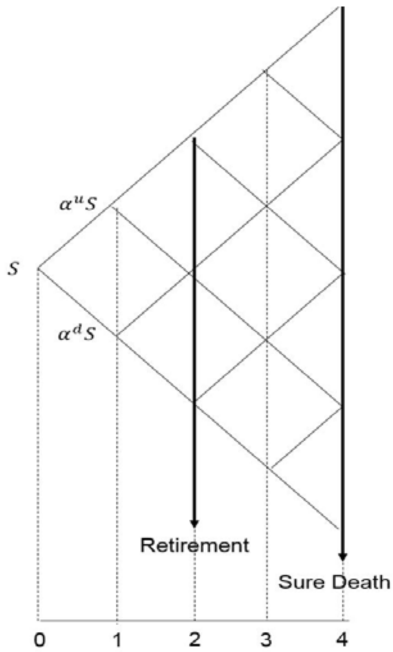

Figure 2. 4-period binomial model with life-time events

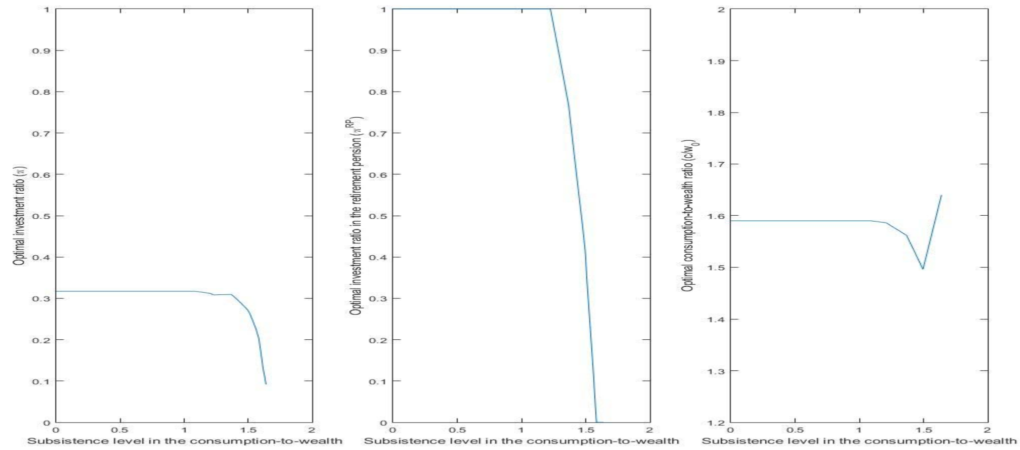

Figure 3. Optimal strategies π^* (left), π^{RP^*} (middle), and C^*/W (right) as functions of subsistence level $\underline{C}^a/W = \underline{C}^b/W$ in the consumption-to-wealth ratio.

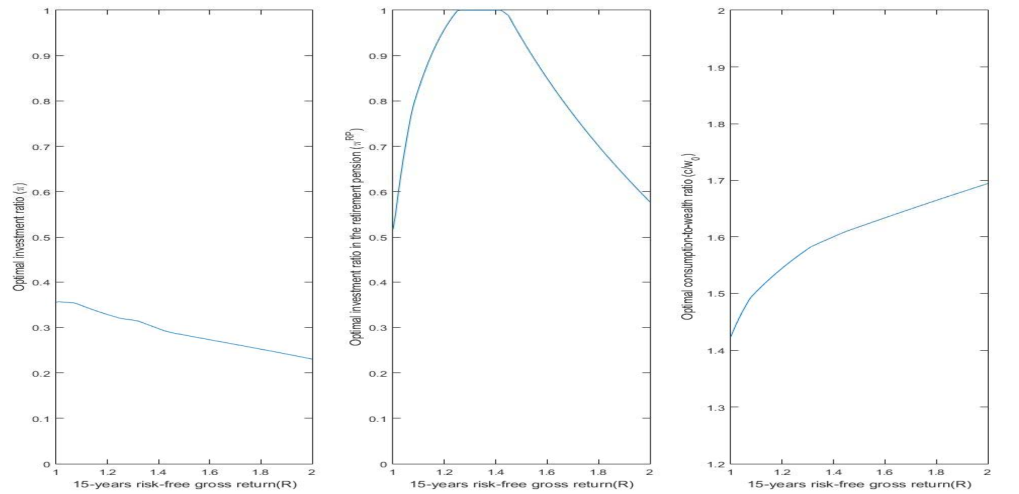

Figure 4. Optimal strategies π^* (left), π^{RP^*}(middle), and C^*/W (right) as functions of risk-free gross return R with subsistence level 1.2 in consumption-to-wealth ratio.

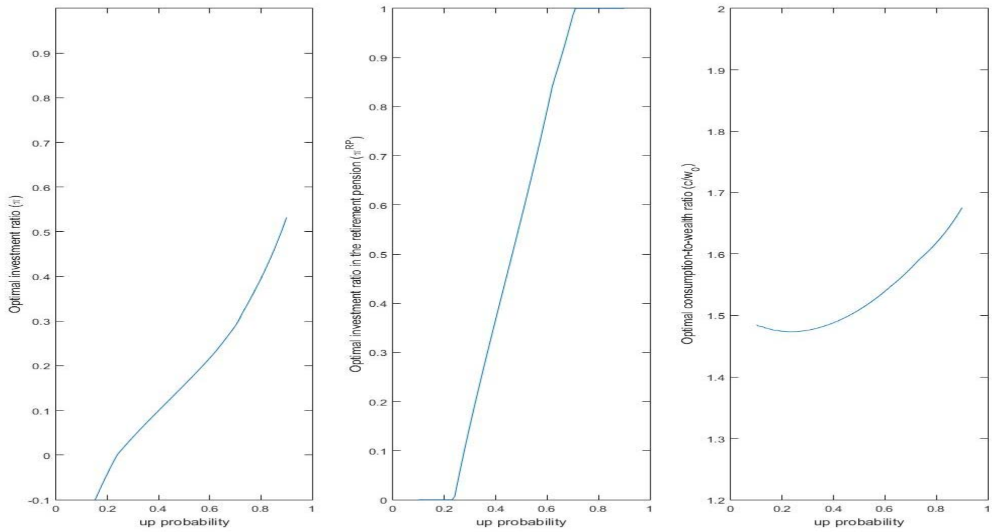

Figure 5. The optimal strategies π^* (left), π^{RP^*} (middle), and C^*/W (right) as functions of up probability p^u with subsistence level 1.2 in consumption-to-wealth ratio.

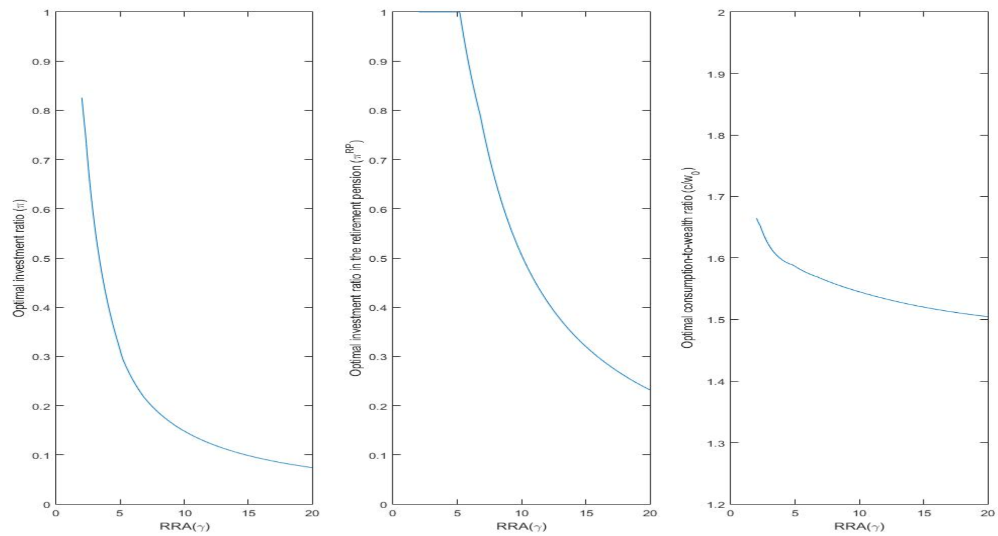

Figure 6. The optimal strategies π^*(left), π^{RP^*}(middle), and C^*/W (right) as functions of the relative risk aversion level γ with subsistence level 1.2 in consumption-to-wealth ratio.

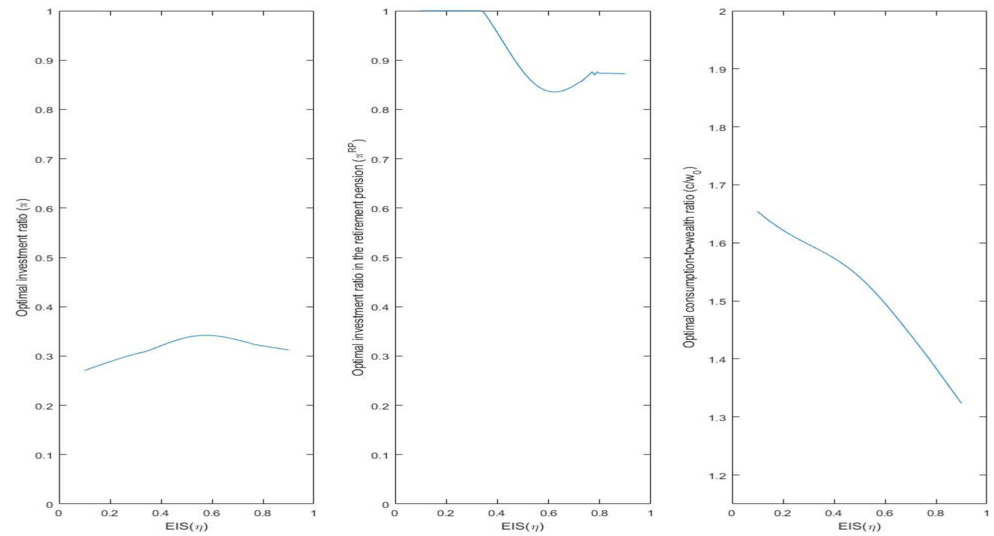

Figure 7. The optimal strategies π^*(left), π^{RP^*}(middle), and C^*/W (right) as functions of the EIS level η with subsistence level 1.2 in consumption-to-wealth ratio.

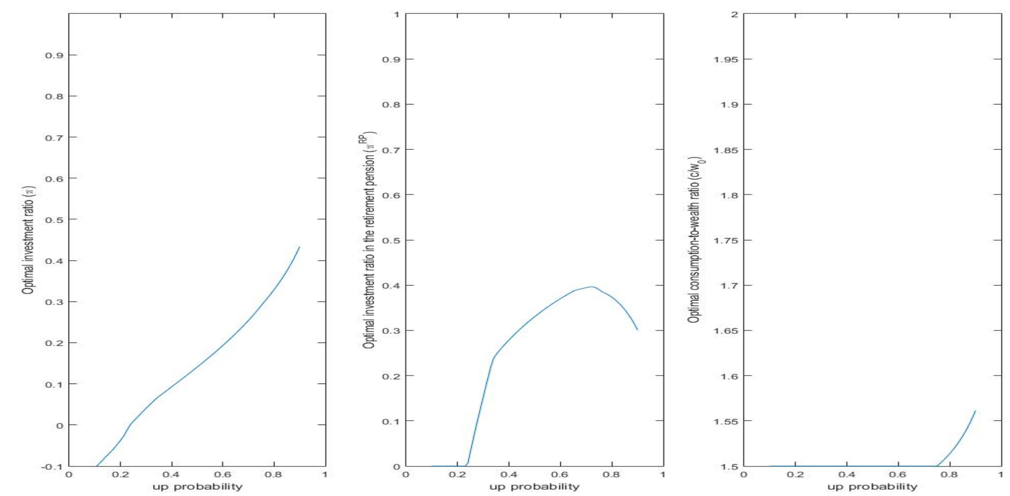

Figure 8. The optimal strategies π^* (left), π^{RP^*} (middle), and C^*/W_0 (right) as functions of up probability p^u with subsistence level 1.5 in consumption-to-wealth ratio.

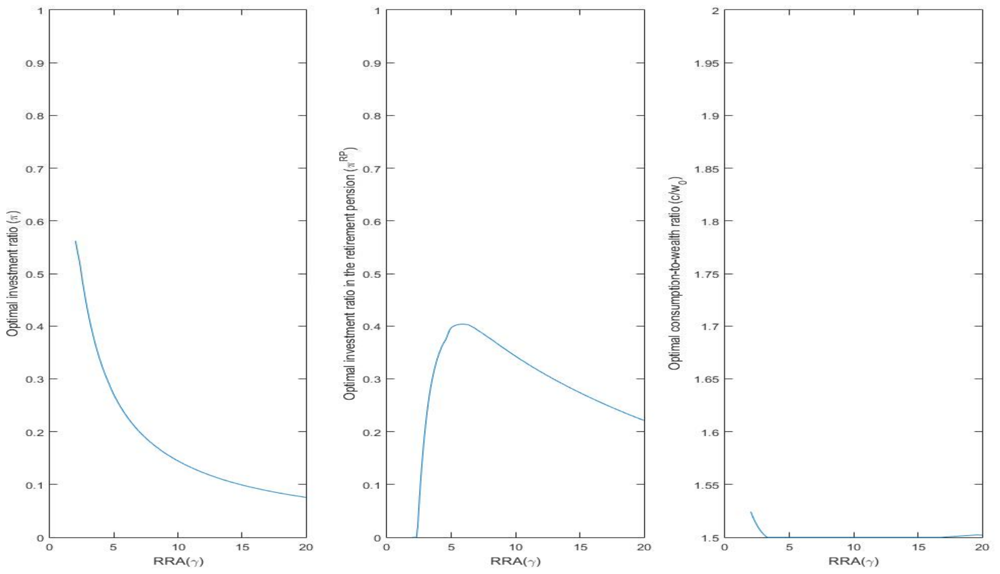

Figure 9. Optimal strategies π^* (left), π^{RP^*} (middle), and C^*/W_0 (right) as functions of the relative risk aversion (RRA) level γ with subsistence level 1.5 in consumption-to-wealth ratio.

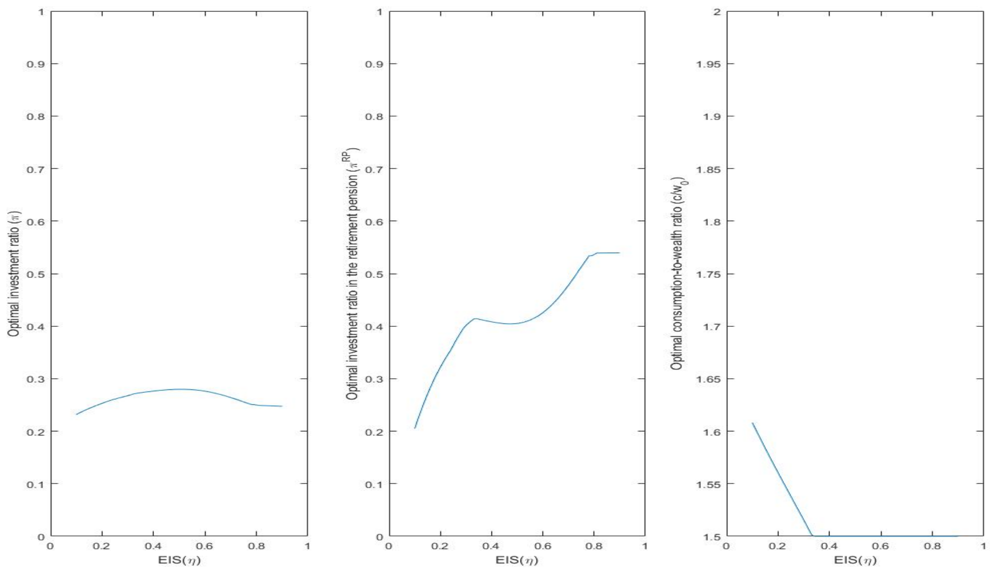

Figure 10. Optimal strategies π^* (left), π^{RP^*} (middle), and C^*/W_0 (right) as functions of the EIS level η with subsistence level 1.5 in consumption-to-wealth ratio.

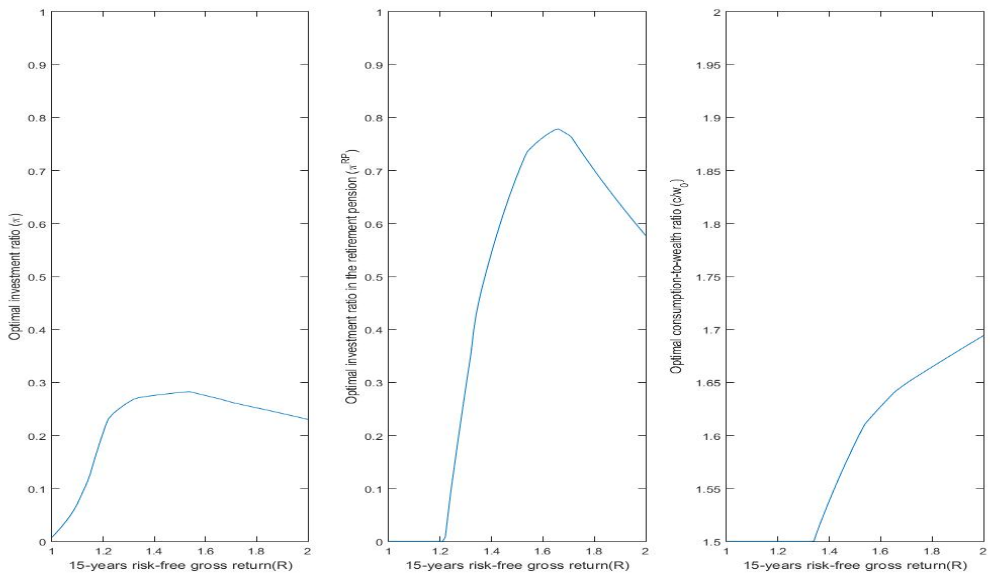

Figure 11. Optimal strategies π^* (left), π^{RP^*} (middle), and C^*/W_0 (right) as functions of risk-free gross return R with subsistence level 1.5 in consumption-to-wealth ratio.

Tables

Table 1. Base parameter set. In this section, we use the parameters in this table. The estimation process is presented in detail in the following paragraphs.

	Parameter	Base Lines
	time interval (Δt, year)	15
Market Conditions	15-year gross return rate of risk free asset (R)	1.33
	probability of up markets (p^u)	0.722719
	15-year gross return rate of risky asset in case of up markets (α^u)	4.28534
	15-year gross return rate of risky asset in case of down markets (α^d)	0.409691
Wealth & Wage	wealth level (W)	\$347200
	wage level ($\{I_0, I_1\}$)	{\$630308, \$644511}
Annuities	portion of wage saved in state social security ($\{\theta_0^S, \theta_1^S\}$)	{7.19%, 10.94%}
	output-to-input ratio of state social security (ρ^S)	43%
	portion of wage saved in retirement pension ($\theta_0^{RP} = \theta_1^{RP}$)	10%
	upper position limit ($\overline{\pi}$)	1.0
	lower position limit ($\underline{\pi}$)	0.0
	mortality rates ($\{\delta_1, \delta_2, \delta_3\} = \{\delta'_1, \delta'_2, \delta'_3\}$)[15]	{4.0%, 13.0%, 38.1%}
Preferences	subjective discount rate (β)	0.86
	level of RRA (γ)	5
	level of EIS (η)	1/3

[15] We assume that the risk-neutral mortality rates are not different from the real mortality rates. We find that the results appeared in the paper are not much different from those in other cases.

Table 2. Estimation results for the parameters related to market conditions. Rows: estimates of stock parameters α^u, α^d, and p^u with different data sets: Ibbotson's Capital Market Assumptions (2nd row), non-overlapped 15-years gross return rate of S&P 500 Index from Yahoo Finance (3rd row), and 15-years gross return of S&P 500 Index with a moving window sampling (4th row).

	Mean	Std.	α^u	α^d	p^u
IA SBBI S&P 500	3.96 $=(1+9.61\%)^{15}$	0.7552 $=0.1950 \times 15^{1/2}$	4.15617	1.05269	0.93679
S&P 500 (1950-2012) Non-overlapping	3.274108	1.732944	4.31	0.375074	0.736744
S&P 500 (1950-2012) Moving Window	3.210693	1.734957	4.28534	0.409691	0.722719