Dynamic Consumption and Portfolio Choice with Permanent Learning*

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Abstract

This paper studies a continuous–time intertemporal consumption and portfolio choice problem when a long–horizon investor does not exactly observe the expected returns of the risky asset. The representative investor who has recursive preferences uses prior belief to estimate the current regime and continuously updates her posterior beliefs with regard to future variation in expected returns. We contribute to solutions to the explicit log-utility case, and to the approximate unit-risk-aversion case. We show explicitly that her belief behavior depends on the parameters of investment opportunities and investor preferences. In addition, the magnitude of the elasticity of intertemporal substitution of consumption determines the relative importance of the substitution and income effects of belief change on consumption.

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1. Introduction

This paper studies how *information quality* affects intertemporal optimal consumption and portfolio choice. In financial economics, information quality related to *parameter learning* represents how updates to information affect asset pricing or investment decisions. Standard portfolio literature (Campbell et al., 2004; Chacko and Viceira, 2005) has commonly assumed that investors have complete knowledge of probability distributions (complete information), so several dynamics that influence their wealth have fixed parameters. However, in practice, these parameters are unknown (incomplete information), so investors must learn them from observed data for use in making forward–looking decisions.

To express this idea precisely, we develop a model in which a long–horizon investor with a recursive utility (Duffie and Epstein, 1992) cannot exactly observe the expected returns of a risky asset; simply, the expected–return process is assumed to follow a two–state and continuous hidden Markov chain. The representative investor learns the time–varying expected returns from risky–asset prices by revising her beliefs when new information arrives.

Specifically, the representative investor has prior belief about the current regime of a financial market, and uses this belief to infer the unknown expected returns. We show that prior belief affects myopic portfolio demand; this short–term investment characteristic is the same for both single–period (myopic) and multi–period investors. At the same time, she continuously updates posterior beliefs with regard to future variation in expected returns. We show that revisions in posterior beliefs governed by a nonlinear mean–reverting stochastic differential equation (SDE) have a *permanent* effect on time variation in investment opportunities.[1] This permanent nature gives rise to *nonlinear* intertemporal hedging demand against adverse shifts of future investment opportunities; this conclusion differs from the linear hedging demand of the standard literature.

Given the rational expectations model, we explore how revisions in posterior beliefs affect

[1] A nonlinear SDE means that the SDE does not have an affine diffusion term with respect to a state variable.

optimal consumption and portfolio policies. In particular, we contribute to solutions to a log-utility case, and to a unit-risk-aversion case.

First, we obtain an explicit solution to the log–utility case: the elasticity of intertemporal substitution (EIS) of consumption and the coefficient of relative risk aversion (RRA) are both equal to one. The conventional wisdom is that a logarithmic investor does behave like a myopic investor. Even though the logarithmic investor focuses only on the short–term investment characteristic, we show that she also updates her posterior beliefs. Her belief behavior depends on the parameters of investment opportunities and investor preferences.

Second, we derive an approximate solution to the unit–risk–aversion case by further relaxing that EIS = 1. The interpretation is similar to the log–utility interpretation in terms of the updating behavior. By contrast, we show that the size of the EIS determines the relative importance of the substitution and income effects of belief change on consumption. The relative importance has a crucial difference on how posterior–belief variation affects the optimal consumption rule.

The rest of the paper is organized as follows. Section 2 states our investment opportunity set and optimization problem. Section 3 studies the optimal policies in response to posterior–belief variation with the two cases. Section 4 concludes.

2. The Intertemporal Consumption and Portfolio Selection

2.1. Investment Opportunity Set

Simply, the investor trades two investment assets. The first one with instantaneous return r is riskless:

$$\frac{dB(t)}{B(t)} = rdt.$$

The second one with time–varying expected return $\mu(t)$ is risky:

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma dZ(t),$$

where the expected return $\mu(t)$ is governed by the two–state hidden Markov chain: high regime H, and low regime L with $\mu_H > \mu_L$; $\sigma > 0$ is a constant standard deviation; and $Z(t)$ is a standard Brownian motion.

The investor does not exactly know the expected return $\mu(t)$ (nor the resulting Brownian motion). To infer the unknown $\mu(t)$, she uses own prior belief $p(t) \equiv \Pr(\mu(t) = \mu_H) \in [0,1]$ at initial time t and then updates posterior beliefs with regard to further variation in expected returns. Given information available at time t, her estimated expected return $\bar{\mu}(t)$ is defined as

$$\bar{\mu}(t) \equiv \mu_H \cdot p(t) + \mu_L(1 - p(t)) = \mu_L + (\mu_H - \mu_L)p(t).$$

The use of a nonlinear filtering theory (Liptser and Shiryaev, 2001) delivers the filtered asset process as

$$\frac{dS(t)}{S(t)} = \bar{\mu}(t)dt + \sigma d\widehat{Z}(t),$$

and the posterior–belief process as

$$\begin{aligned} dp(t) &= \{\lambda_L - (\lambda_H + \lambda_L)p(t)\}dt + \frac{\mu_H - \mu_L}{\sigma}p(t)(1 - p(t))d\widehat{Z}(t) \\ &\equiv \mu_p(p(t))dt + \sigma_p(p(t))d\widehat{Z}(t), \end{aligned} \tag{1}$$

where λ_i for $i, j \in \{H, L\}$ is a transition intensity such that regime i jumps to regime j for $j \notin i$, and the new filtered Brownian motion $\widehat{Z}(t)$ is defined as

$$\widehat{Z}(t) = \int_0^t \frac{dS(u) - \bar{\mu}(u)S(u)du}{\sigma S(u)}.$$

Our investment opportunity set describes her wealth process $W(t)$ evolving as

$$\begin{aligned} dW(t) &= \big[\{r + \pi(t)(\bar{\mu}(t) - r)\}W(t) - C(t)\big]dt + \sigma\pi(t)W(t)d\widehat{Z}(t) \\ &\equiv \mu_w(W(t), p(t), C(t), \pi(t))dt + \sigma_w(W(t), \pi(t))d\widehat{Z}(t), \end{aligned} \tag{2}$$

where $C(t)$ denotes consumption, and $\pi(t)$ is the proportion of wealth invested in the risky asset.

Our investment opportunity set differs from the set used in Campbell et al. (2004). First, our expected–return process $d\bar{\mu}(t) = (\mu_H - \mu_L)dp(t)$ includes the concave diffusion term $(\mu_H - \mu_L)\sigma_p \equiv \frac{(\mu_H-\mu_L)^2}{\sigma} p(t)(1-p(t))$ that depends on prior belief, whereas Campbell et al. (2004) consider the constant diffusion case. Second, we study a complete–market case, in which shocks to $d\bar{\mu}(t)$ are perfectly correlated with shocks to $dS(t)/S(t)$. By contrast, Campbell et al. (2004) study an incomplete–market case, in which shocks to $d\mu(t)$ are *not* perfectly correlated with shocks to $dS(t)/S(t)$. Third, our investment opportunity set characterizes momentum investors; they learn a trend in expected–return variation from risky asset's observations. To see this, consider the covariance between $dS(t)/S(t)$ and $d\bar{\mu}(t)$:

$$\text{Cov}(dS(t)/S(t), d\bar{\mu}(t)) = (\mu_H - \mu_L)^2 p(t)(1-p(t)) \geq 0.$$

Except for perfect beliefs ($p(t) = 0$ or 1), the positive covariance demonstrates the momentum property. This contrasts with Campbell et al. (2004), who consider that $\text{Cov}(dS(t)/S(t), d\mu(t)) < 0$. In sum, our purpose is to study how this learning characteristic affects optimal consumption and investment policies.

2.2. Investor Preferences and Optimization Problem

The investor wants to maximize her utility function by controlling consumption and investment rules subject to the budget constraint (2). Specifically, her preferences over consumption are represented by a continuous–time recursive utility (Duffie and Epstein, 1992):

$$J = E_t \left[\int_t^{\infty} f(C(s), J(s)) ds \right], \tag{3}$$

where $E_t[\cdot]$ denotes the expectation given information available at initial time t, and $f(C(t), J(t))$ is a normalized aggregator of current consumption and continuation utility:

$$f(C(t), J(t)) = \frac{\beta}{1-1/\psi}(1-\gamma)J(t)\left[\left\{\frac{C(t)}{((1-\gamma)J(t))^{1/(1-\gamma)}}\right\}^{(1-1/\psi)} - 1\right], \tag{4}$$

where β is the rate of time preference. As $\psi \to 1$, this normalized aggregator becomes

$$f(C(t), J(t)) = \beta(1-\gamma)J(t)\left[\log(C(t)) - \frac{1}{1-\gamma}\log((1-\gamma)J(t))\right]. \tag{5}$$

The recursive–utility function has a merit in that it can separate the RRA γ from the EIS ψ; here, the case of $\gamma = 1/\psi$ corresponds to the standard power-utility function.

The optimization problems in (1)–(5) depend on the investor's wealth and posterior belief. The resulting Bellman equation is

$$0 = \sup_{\{C(t),\pi(t)\}}\left[f(C(t), J(t)) + \mu_w J_W + \frac{1}{2}\sigma_w^2 J_{WW} + \mu_p J_p + \frac{1}{2}\sigma_p^2 J_{pp} + \sigma_p\sigma_w J_{Wp}\right], \tag{6}$$

where J_x denotes the partial derivative of the value function J with respect to subscript x. The first–order conditions with respect to consumption and portfolio rules are

$$\begin{aligned} C(t) &= J_W^{-\psi}[(1-\gamma)J]^{(1-\gamma\psi)/(1-\gamma)}\beta^{\psi}, \quad \text{and} \\ \pi(t) &= -\frac{J_W}{W(t)J_{WW}}\left\{\frac{\bar{\mu}(t)-r}{\sigma^2}\right\} - \frac{J_{Wp}}{W(t)J_{WW}}p(t)(1-p(t))\left(\frac{\mu_H-\mu_L}{\sigma^2}\right). \end{aligned} \tag{7}$$

To extract economic insights, we will guess some forms of the value function J in the next section.

3. Optimal Policies

In this section, we study optimal consumption and portfolio rules when a rational investor faces the investment opportunity set and some preferences discussed in Section 2. Specifically, we study the log–utility case ($\gamma = 1/\psi = 1$) in Section 3.1, and the unit–RRA case ($\gamma = 1$) in Section 3.2.

3.1. A Solution with Unit EIS and Unit RRA

Following the standard literature, the value function $J(W(t), p(t))$ has the form

$$J(W(t), p(t)) = I(p(t))\frac{W(t)^{1-\gamma}}{1-\gamma} \equiv \exp\{(1-\gamma)g(p(t))\}\frac{W(t)^{1-\gamma}}{1-\gamma}. \tag{8}$$

The form of $I(p(t))$ follows from the aggregator $f(C(t), J(t))$ (5) that J is homogeneous of degree $1-\gamma$ in the level of consumption. Here, we assume that the function $g(p(t))$ is twice continuously differentiable with respect to prior belief $p(t)$. Now, $g(p(t))$ is interpreted as investor's belief behavior that reflects future revisions in posterior beliefs.

Proposition 1. *For an investor with the unit EIS ($\psi = 1$), the value function (8) corresponds to the following optimal consumption and portfolio rules:*

$$\frac{C(t)}{W(t)} = \beta, \ \textit{and} \tag{9}$$

$$\pi(t) = \left(\frac{1}{\gamma}\right)\left\{\frac{\bar{\mu}(t) - r}{\sigma^2}\right\} + \left(1 - \frac{1}{\gamma}\right)\left(\frac{\mu_H - \mu_L}{\sigma^2}\right)p(t)(1-p(t))\{-g_p(p(t))\}, \tag{10}$$

where $g_p(p(t))$ is the first–order derivative of the belief behavior $g(p(t))$ with respect to prior belief $p(t)$.

Proof: The proof is obtained by substituting the partial derivatives of J into the optimal policies (7).

Eq. (9) shows that optimal consumption–wealth ratio $C(t)/W(t)$ is myopic. The myopic consumption rule follows that $C(t)/W(t)$ is equal to the constant time–preference rate β; this finding is consistent with the standard literature. In short, when $\psi = 1$, the income and substitution effects of belief change on consumption offset each other *exactly*.

Eq. (10) shows that optimal portfolio choice $\pi(t)$ is *not* myopic. This portfolio choice represents a weighted average of myopic portfolio demand with a weight $1/\gamma$ (the first term) and Mertons' intertemporal hedging demand with a weight $1 - 1/\gamma$ (the second term). In particular, because $\bar{\mu}(t) \equiv \mu_L + (\mu_H - \mu_L)p(t)$, the myopic demand is an increasing monotonic function of prior belief $p(t)$. By contrast, the hedging demand is surely *nonlinear* as a result of the belief–diffusion term $\frac{\sigma_p}{\sigma} \equiv \frac{\mu_H - \mu_L}{\sigma^2} p(t)(1-p(t))$ being multiplied by the marginal–belief behavior $g_p \equiv g_p(p(t))$.

The direct analysis of the hedging demand with respect to γ is a difficult task because we still do not know the form of g_p. To simply explain this difficulty, we substitute the optimal policies in Proposition 1 into the Bellman equation (6):

$$0 = \beta \log \beta - \beta + r + \frac{\vartheta(t)^2}{2\gamma} - \beta g + \frac{1}{2}\sigma_p^2 g_{pp} + \left[\mu_p - \left(1 - \frac{1}{\gamma}\right)\sigma_p \vartheta(t)\right] g_p - \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\sigma_p^2 g_p^2, \quad (11)$$

where $\vartheta(t) \equiv (\bar{\mu}(t) - r)/\sigma$ is defined as the *filtered Sharpe ratio* of the financial market. The second–order ordinary differential equation (ODE) (11) reduced from the Bellman equation (6) is possibly degenerate due to σ_p at the perfect beliefs, and is also nonlinear due to the squared marginal–belief behavior g_p^2. A general solution to the degenerate and nonlinear ODE seems almost impossible. This task would require intensive numerical analysis, and is beyond the scope of this paper, but we find easily that γ has a *clear* effect on intertemporal hedging demand.

Eliminating the hedging demand yields three analytic cases: (i) when investors have the log utility ($\gamma = 1/\psi = 1$), (ii) when investors have perfect beliefs ($\sigma_p = 0$), and (iii) when the marginal–belief behavior becomes zero ($g_p = 0$). All three cases indicate that multi–period investors behave like myopic investors.

To overcome the analytic difficulty, we focus on the logarithmic case (i), because it can provide a benchmark for conjectures regarding the belief behavior of risk–averse investors who have $\gamma > 1$ and of risk–seeking investors who have $\gamma < 1$. The size of γ crucially affects the magnitude of the marginal utility of consumption relative to the benchmark. Although the logarithmic investor only has myopic portfolio demand, we argue that she should be also concerned about future variation in expected returns governed by posterior beliefs. To see this, we substitute $\gamma = 1$ into the ODE (11), which results in the linear (but degenerate) ODE:

$$0 = \beta \log \beta - \beta + r + \frac{\vartheta(t)^2}{2} - \beta g + \frac{1}{2}\sigma_p^2 g_{pp} + \mu_p g_p.$$

The existence of the ODE justifies our argument, provided that the belief behavior g is not constant. Now, we will study logarithmic investor's belief behavior $g(p(t))$.

Proposition 2. *A logarithmic investor with $\gamma = 1/\psi = 1$ has the explicit belief behavior*

$$\begin{aligned} g(p(t)) &= E_t\left[\int_t^\infty e^{-\beta(s-t)}\left\{\beta \log \beta - \beta + r + \frac{1}{2}\vartheta(s)^2\right\} ds\right] \\ &= \log \beta - 1 + \frac{r}{\beta} + \frac{1}{2}E_t\left[\int_t^\infty e^{-\beta(s-t)}\vartheta(s)^2 ds\right]. \end{aligned}$$

Proof: Apply an infinite version of the Feynman–Kac formula (see Theorem 3.5.3 and Remark 3.5.6 of Pham (2009)).

Proposition 2 shows that the belief behavior $g \equiv g(p(t))$ is a function of prior belief $p(t)$; a solution to SDEs follows a Markov process (Shreve, 2004).[2] However, the guess form $I(p(t)) \equiv \exp\{(1-\gamma)g(p(t))\}$ in the value function (8) suggests that posterior–belief variation does not affect a log value function. This finding is analogous to those of the standard literature because logarithmic investors behave like myopic investors. Next, we explore the marginal–belief behavior $g_p(p(t))$ that is an important source of intertemporal hedging demand (10).

Proposition 3. *A logarithmic investor with $\gamma = 1/\psi = 1$ has the following marginal–belief behavior*

$$g_p(p(t)) = \left(\frac{\mu_H - \mu_L}{\sigma}\right) E_t\left[\int_t^\infty e^{-\beta(s-t)}\vartheta(s) I(s) ds\right],$$

where $I(s) \equiv \partial p(s)/\partial p(t)$, and the SDE of $I(s)$ follows

$$dI(s) = -(\lambda_H + \lambda_L)I(s)ds + (1 - 2p(s))\frac{\mu_H - \mu_L}{\sigma} I(s) d\widehat{Z}(s).$$

[2] The fact that the belief process (1) satisfies the Lipschitz and growth conditions, so it can justify a unique solution to $p(t)$, although we cannot obtain the explicit distribution of $p(t)$.

Proof: Apply Malliavin calculus to the belief behavior $g(p(t))$.[3]

Proposition 3 shows that the sign of $g_p \equiv g_p(p(t))$ depends *solely* on the filtered Sharpe ratio $\vartheta(s) \equiv (\bar{\mu}(s) - r)/\sigma$ for time $s \in [t, \infty]$. This result occurs because $I(t) = 1$; i.e., $I(s) > 0$ at any time s. However, we do not guarantee the sign of the second–order marginal–belief behavior g_{pp} that determines whether the belief behavior g is convex or concave:

$$g_{pp}(p(t)) = \left(\frac{\mu_H - \mu_L}{\sigma}\right)^2 E_t\left[\int_t^\infty e^{-\beta(s-t)} I(s)^2 ds\right] + \left(\frac{\mu_H - \mu_L}{\sigma}\right) E_t\left[\int_t^\infty e^{-\beta(s-t)} \vartheta(s) J(s) ds\right],$$

where $J(s) \equiv \partial I(s)/\partial p(t)$, and the SDE of $J(s)$ evolves as

$$dJ(s) = -(\lambda_H + \lambda_L) J(s) ds + \left(\frac{\mu_H - \mu_L}{\sigma}\right) \left\{(1 - 2p(s)) J(s) - 2I(s)^2\right\} d\widehat{Z}(s).$$

The fact of $J(t) = 0$ shows that $g_{pp} \equiv g_{pp}(p(t))$ is not always positive. Thus, the curvature depends on the parameters of the investment opportunity set.

Corollary 3.1. *If $\mu_L \geq r$ for all time $s \in [t, \infty]$, logarithmic investor's belief behavior $g(p(t))$ is an increasing function with respect to prior belief $p(t)$.*

Corollary 3.1 is intuitive because the condition of $\mu_L \geq r$ excludes short–sale availability.[4] In general, the worst–case Sharpe ratio $\vartheta_L \equiv (\mu_L - r)/\sigma$ becomes positive so that all the ratios $\vartheta(s)$ for any time $s \in [t, \infty]$ are consistently positive. As a result, the product of $\vartheta(s) \geq 0$ and $I(s) > 0$ in Proposition 3 shows that $g_p \geq 0$; so showing Corollary 3.1 suffices.

3.2. An Approximate Solution with Unit RRA

Following the standard literature, the value function $J(W(t), p(t))$ has the form

$$J(W(t), p(t)) = I(p(t)) \frac{W(t)^{1-\gamma}}{1-\gamma} = \exp\left\{\frac{1-\gamma}{1-\psi} \cdot h(p(t))\right\} \frac{W(t)^{1-\gamma}}{1-\gamma}. \tag{12}$$

[3] See Honda (2003) for more technical details.

[4] Suppose $\vartheta(t) = 0$. This inspection gives $p^*(t) = (r - \mu_L)/(\mu_H - \mu_L) \in [0, 1]$. If $\mu_L < r$, short sales are optimal over some priors to avail the worst market condition.

Here, we use the general aggregator $f(C(t), J(t))$ (4), not (5) that was used in Section 3.1.

Proposition 4. *For an investor who does not have the unit EIS ($\psi \neq 1$), the value function (12) corresponds to the following optimal consumption and portfolio rules:*

$$\frac{C(t)}{W(t)} = \beta^{\psi} \exp\{h(p(t))\}, \text{ and} \tag{13}$$

$$\pi(t) = \left(\frac{1}{\gamma}\right)\left\{\frac{\bar{\mu}(t) - r}{\sigma^2}\right\} + \left(1 - \frac{1}{\gamma}\right)\left(\frac{\mu_H - \mu_L}{\sigma^2}\right) p(t)(1 - p(t)) \left\{-\frac{h_p(p(t))}{1 - \psi}\right\}, \tag{14}$$

where $h_p(p(t))$ is the first–order derivative of the belief behavior $h(p(t))$ with respect to prior belief $p(t)$.

Proof: Simply substitute the partial derivatives of J into the optimal policies (7).

Eq. (13) shows that log optimal consumption–wealth ratio $C(t)/W(t)$ is an affine function of the belief behavior $h \equiv h(p(t))$ along with ψ and β. Because we do not know the explicit belief–behavior form, we attempt to find approximate belief behavior later.

Eq. (14) shows that optimal portfolio choice consists of myopic portfolio demand (the first term) and intertemporal hedging demand (the second term). This portfolio composition is similar to that when $\psi = 1$ (Proposition 1), but the marginal–belief behavior $h_p \equiv h_p(p(t))$ is further scaled by $1/(1 - \psi)$. When multi–period investors focus only on myopic portfolio demand, three analytic cases arise: (i) when investors have the unit RRA ($\gamma = 1$), (ii) when investors have perfect beliefs ($\sigma_p = 0$), and (iii) when the marginal–belief behavior becomes zero ($h_p = 0$).

To help interpret Proposition 4, especially for the optimal consumption rule, we substitute the optimal policies in Proposition 4 into the Bellman equation (6):

$$\begin{aligned} 0 = r(1-\psi) + \beta\psi + \frac{1-\psi}{2\gamma}\vartheta(t)^2 - \beta^{\psi}\exp\{h\} + \frac{1}{2}\sigma_p^2 h_{pp} + \left[\mu_p - \left(1 - \frac{1}{\gamma}\right)\sigma_p\vartheta(t)\right] h_p \\ - \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\left(\frac{1}{1-\psi}\right)\sigma_p^2 h_p^2. \end{aligned} \tag{15}$$

The second–order ODE (15) reduced from the Bellman equation (6) is possibly degenerate and is also nonlinear. To help understand the ODE (15), we further linearize the optimal consumption–wealth ratio (13) around the log long–term mean of $\overline{c-w} \equiv E[\log\{C(t)/W(t)\}]$ by using a first–order Taylor expansion

$$\frac{C(t)}{W(t)} \approx \kappa_0 + \kappa_1(\psi \log \beta + h(p(t))), \tag{16}$$

where $\kappa_1 = \exp\{\overline{c-w}\}$ and $\kappa_0 = \kappa_1(1 - \log \kappa_1)$.[5] If the ratio (16) does not deviate seriously from the long–term mean, this approximate ratio should be similar to the real ratio (13).

The substitution of the approximate ratio (16) into the ODE (15) yields

$$\begin{aligned} 0 = r(1-\psi) + \psi(\beta - \kappa_1 \log \beta) - \kappa_0 + \frac{1-\psi}{2\gamma}\vartheta(t)^2 - \kappa_1 \cdot h + \frac{1}{2}\sigma_p^2 h_{pp} + \left[\mu_p - \left(1 - \frac{1}{\gamma}\right)\sigma_p \vartheta(t)\right] h_p \\ - \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\left(\frac{1}{1-\psi}\right)\sigma_p^2 h_p^2. \end{aligned} \tag{17}$$

An approximate solution to the resulting ODE (17) still seems difficult to solve. Accordingly, we focus on the case where $\gamma = 1$. This benchmark case leads to the following approximate (but degenerate) ODE:

$$0 = r(1-\psi) + \psi(\beta - \kappa_1 \log \beta) - \kappa_0 + \frac{1-\psi}{2}\vartheta(t)^2 - \kappa_1 \cdot h + \frac{1}{2}\sigma_p^2 h_{pp} + \mu_p h_p. \tag{18}$$

The existence of the ODE (18) can demonstrate that even the benchmark investor with $\gamma = 1$ continuously updates her posterior beliefs in response to risky asset's behavior, unless h is constant. Now, we will study benchmark belief behavior $h(p(t))$.

Proposition 5. *An investor with the unit RRA ($\gamma = 1$) has the explicit belief behavior*

$$\begin{aligned} h(p(t)) &= E_t\left[\int_t^\infty e^{-\kappa_1(s-t)}\left\{r(1-\psi) + \psi(\beta - \kappa_1 \log \beta) - \kappa_0 - \kappa_1 h + \frac{1-\psi}{2}\vartheta(s)^2\right\} ds\right] \\ &= \frac{r}{\kappa_1}(1-\psi) + \psi\left(\frac{\beta}{\kappa_1} - \log \beta\right) - \frac{\kappa_0}{\kappa_1} + \frac{1-\psi}{2} E_t\left[\int_t^\infty e^{-\kappa_1(s-t)}\vartheta(s)^2 ds\right]. \end{aligned}$$

Proof: Apply an infinite version of the Feynman–Kac formula (Pham, 2009).

[5] See Campbell et al. (2004) and Chacko and Viceira (2005) for the log–linear expansion.

Proposition 5 suggests that the belief behavior $h \equiv h(p(t))$ is a general version of the logarithmic behavior $g \equiv g(p(t))$ by relaxing the constraint that $\psi = 1$. To clarify this relationship, consider the limit of the optimal consumption–wealth ratio, i.e., $\lim_{\psi \to 1} C(t)/W(t)$. As $\psi \to 1$ the consumption–wealth ratio $C(t)/W(t)$ approaches the time–preference rate β (Section 3.1); the constant ratio leads to $\kappa_1 = \beta$ and $\kappa_0 = \beta(1 - \log \beta)$. Given that the two value functions in (12) and (8) are related as $h = (1-\psi)g$, all parameterizations correspond to the logarithmic case:

$$\lim_{\psi \to 1} \frac{h(p(t))}{1-\psi} = g(p(t)).$$

The guess form $I(p(t)) \equiv \exp\{\frac{1-\gamma}{1-\psi} h(p(t))\}$ also suggests that revisions in posterior beliefs with $\gamma = 1$ do not affect the value function (12); this finding is analogous to the finding in Section 3.1.

Importantly, the size of ψ is the important determinant of whether the belief behavior h is increasing or decreasing. To see this, we differentiate the belief behavior h with respect to prior belief:

$$h_p(p(t)) = (1-\psi) \left(\frac{\mu_H - \mu_L}{\sigma} \right) E_t \left[\int_t^{\infty} e^{-\kappa_1 (s-t)} \vartheta(s) I(s) ds \right]. \tag{19}$$

Given Corollary 3.1, if $\psi < 1$, the belief behavior h is increasing in prior belief $p(t)$ with the Markov property; if $\psi > 1$, h is decreasing.

Now, we study the effect of ψ on the optimal consumption rule (13) for the benchmark investor with $\gamma = 1$. Note that an increase in prior belief has two conflicting effects on consumption. On one hand, the increased belief causes a positive substitution effect of belief change on consumption; that is, she increases consumption today. She responds in this way because the increased belief improves investment opportunities (see the Sharpe ratio $\vartheta(t)$). On the other hand, the increased belief also causes a negative income effect on consumption; that is, she decreases her consumption level relative to total wealth. This decrease follows that she would anticipate the subsequent (downward) mean reversion of the risky asset in the future; the mean reversion driven by revisions in posterior beliefs increases the marginal utility of consumption. By contrast, the decreased belief induces the opposite consumption reaction: i.e., a negative substitution effect, and a positive income effect.

The net effect with $\gamma = 1$ on consumption distinguishes investors with $\psi < 1$ from those with $\psi > 1$. To understand how this distinction is made, see the approximate consumption–wealth ratio (16). Because the case of $\psi < 1$ delivers an increasing belief function with respect to prior belief (see (19)), investors with $\psi < 1$ increase consumption as prior belief increases. This finding implies that the substitution effect dominates the income effect; these investors focus more on the short–term investment characteristic than on the adverse mean–reverting shifts of future investment opportunities. By contrast, investors with $\psi > 1$ decreases consumption against potential deterioration in investment opportunities. The similar argument demonstrates that the income effect dominates the substitution effect; these investors put more emphasis on potential loss against future variation in expected returns than on the short–term improvement in investment opportunities.

4. Conclusion

The standard literature has presumed that investors have perfect knowledge of price movements. However, in practice, investors are not confident about the true price behavior. Potential misjudgment may induce erroneous consumption and portfolio decisions.

In this paper, we postulate that the financial market is governed by the two–state hidden Markov chain and focus on two benchmark cases. We show that information quality has a significant effect on investors' belief behavior; it depends clearly on the parameters that characterize investment opportunities and investor preferences.

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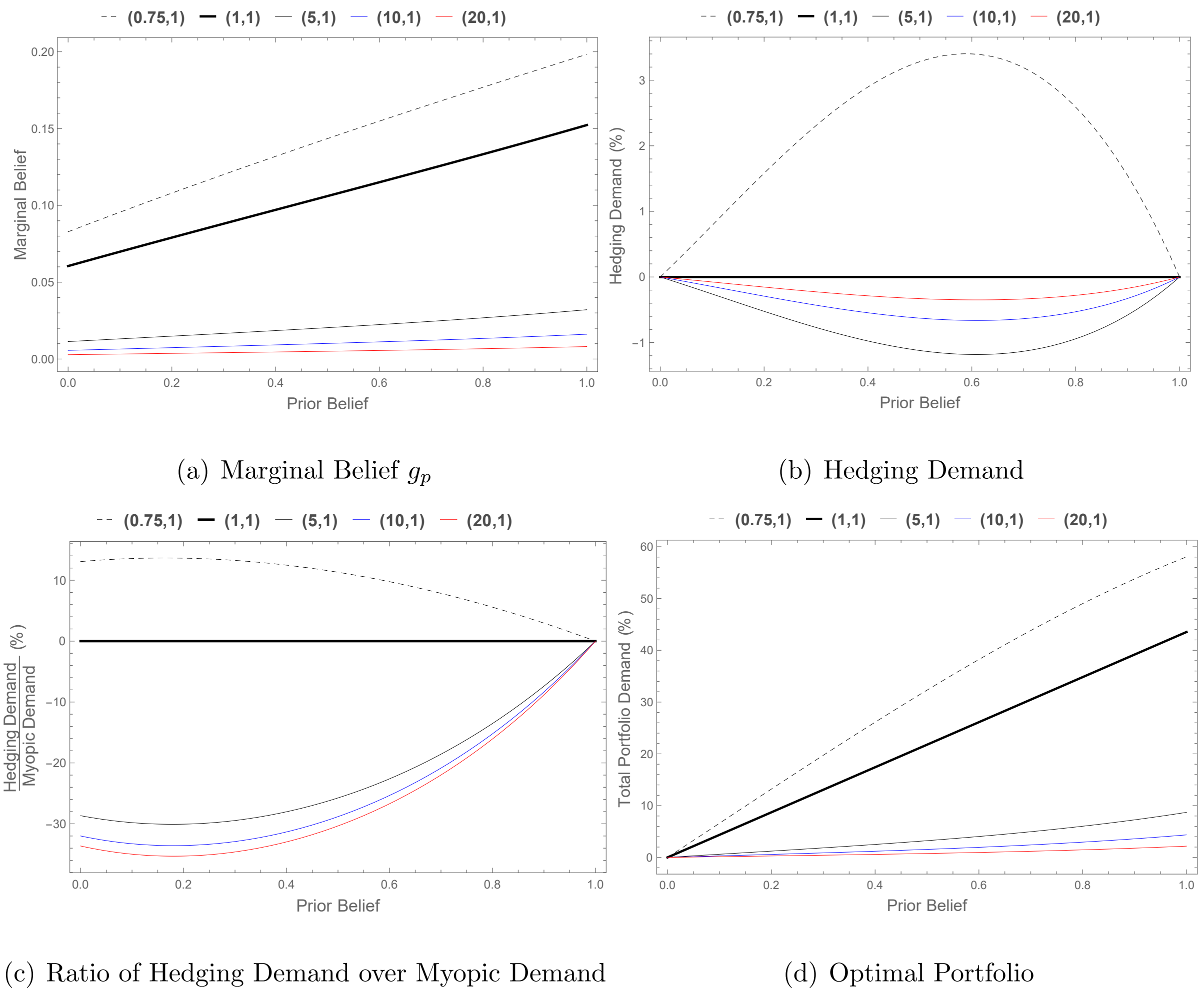
Supplementary Materials

We estimate annual real (inflation–adjusted) parameters of the investment opportunities and smoothed probabilities $p(t)$ over a sample period from 1947 to 2015, given Corollary 3.1.[6] Then, we use the Judd's (1998) projection method to obtain the numerical results in Sections 3.1 and 3.2.[7]

Section 3.1: A Solution with Unit EIS



(a) Marginal Belief g_p

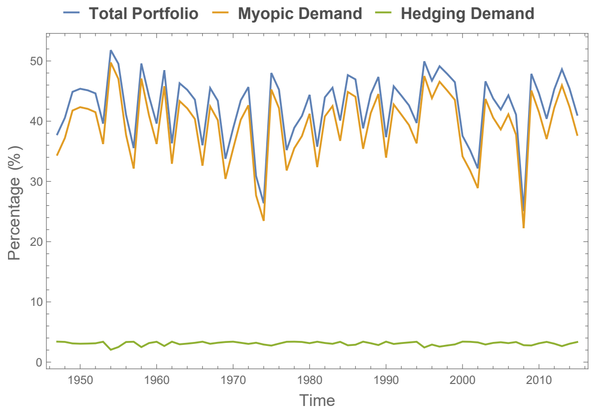
(b) Hedging Demand

(c) Ratio of Hedging Demand over Myopic Demand

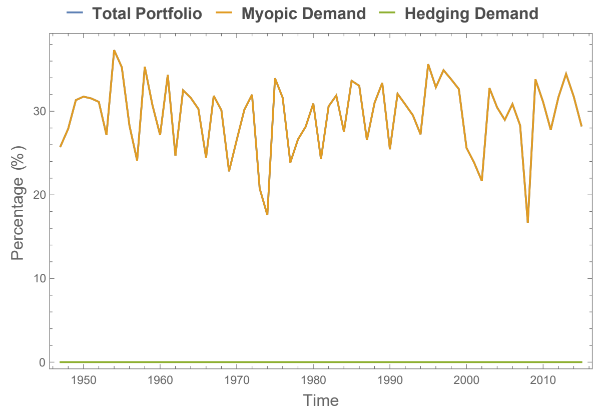
(d) Optimal Portfolio

Figure 1: Optimal Portfolio Policy with (γ, ψ). The annual estimated parameters are $\mu_H = 0.0757$, $\mu_L = r$, $\lambda_H = 0.3155$, $\lambda_L = 0.6667$, $r = 0.0067$, $\sigma = 0.1586$, and $\beta = 0.06$.

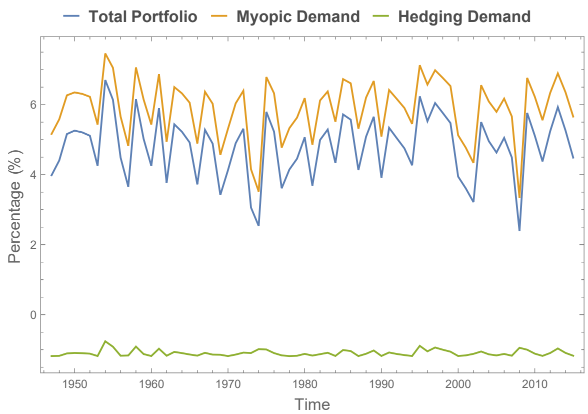
[6] We use the data from Robert J. Shiller's web site, " http://www.econ.yale.edu/ shiller/data.htm".
[7] Judd, K.L., 1998. Numerical methods in economics. The MIT Press.



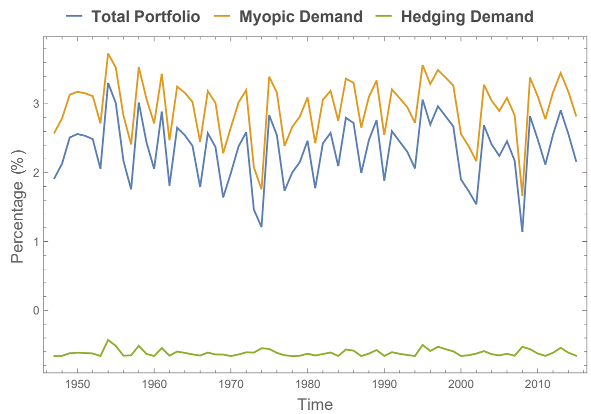
(a) Time–Series Portfolio $(0.75, 1)$



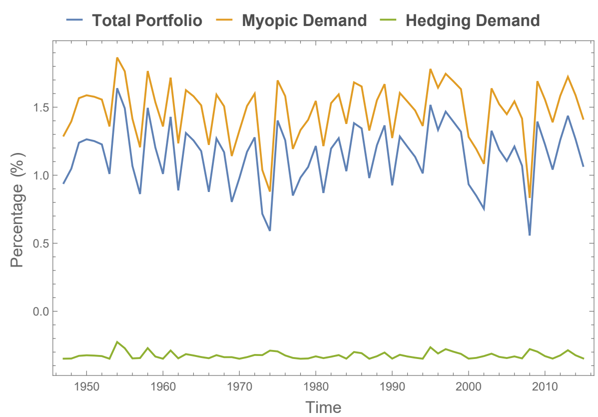
(b) Time–Series Portfolio $(1, 1)$



(c) Time–Series Portfolio $(5, 1)$



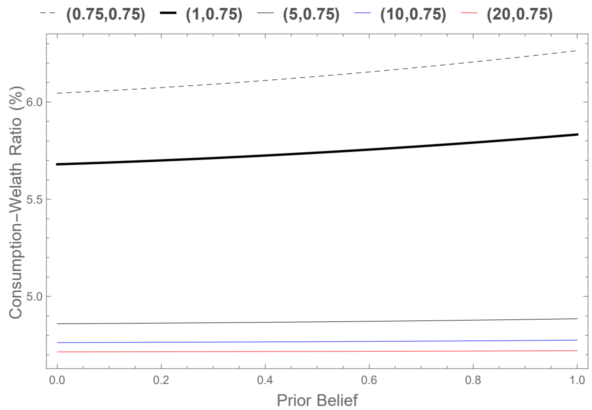
(d) Time–Series Portfolio $(10, 1)$



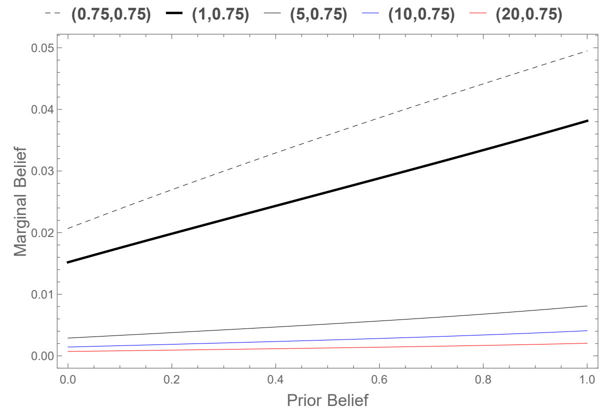
(e) Time–Series Portfolio $(20, 1)$

Figure 2: Implied Time–Series Portfolio Proportion with (γ, ψ). The annual estimated parameters are $\mu_H = 0.0757$, $\mu_L = r$, $\lambda_H = 0.3155$, $\lambda_L = 0.6667$, $r = 0.0067$, $\sigma = 0.1586$, and $\beta = 0.06$.

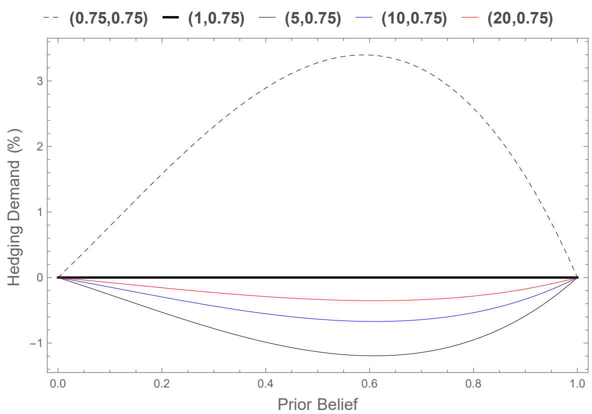
Section 3.2: An Approximate Solution with $EIS = 0.75 < 1$



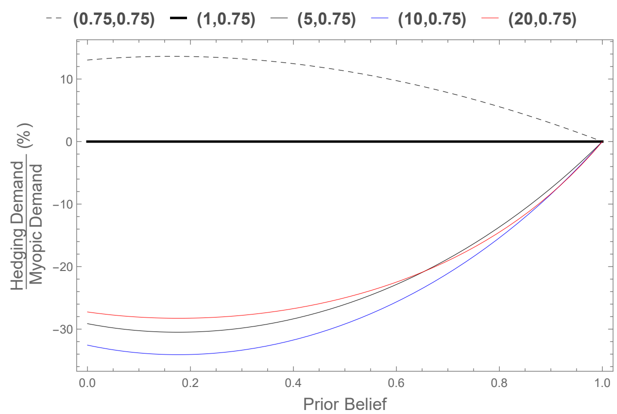
(a) Consumption–Wealth Ratio



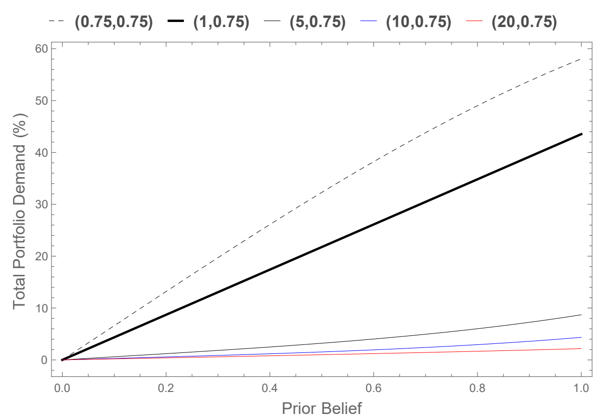
(b) Marginal Belief h_p



(c) Hedging Demand

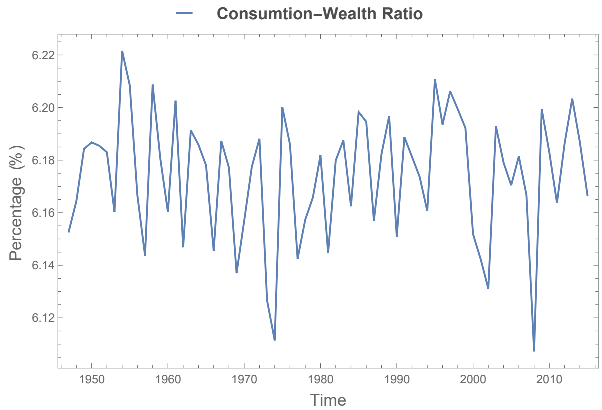


(d) Ratio of Hedging Demand over Myopic Demand

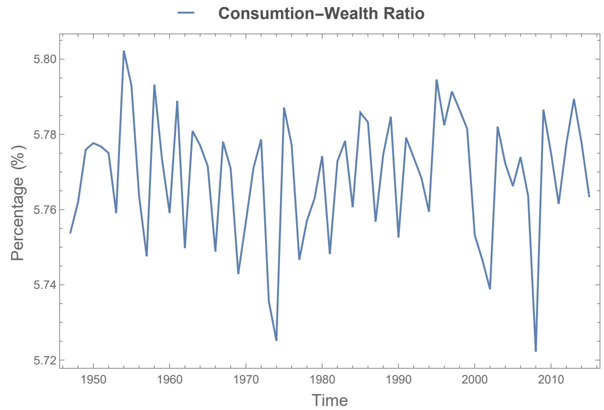


(e) Optimal Portfolio

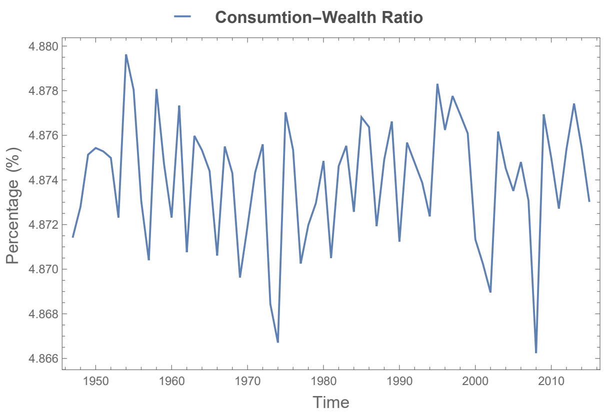
Figure 3: Optimal Consumption and Portfolio Policies with (γ, ψ). The annual estimated parameters are $\mu_H = 0.0757$, $\mu_L = r$, $\lambda_H = 0.3155$, $\lambda_L = 0.6667$, $r = 0.0067$, $\sigma = 0.1586$, and $\beta = 0.06$.



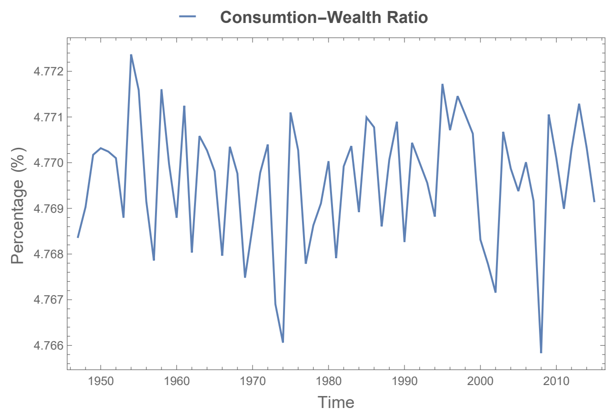
(a) Time–Series Consumption $(0.75, 0.75)$



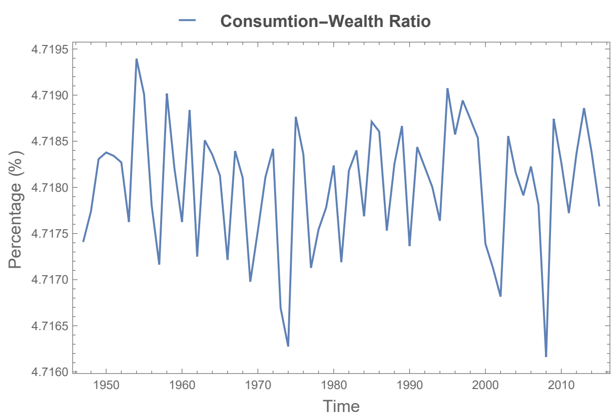
(b) Time–Series Consumption $(1, 0.75)$



(c) Time–Series Consumption $(5, 0.75)$

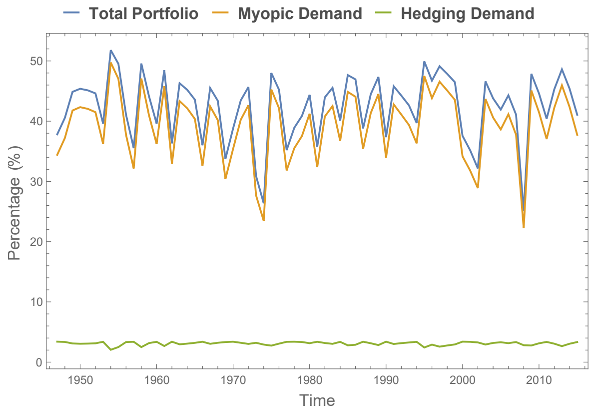


(d) Time–Series Consumption $(10, 0.75)$

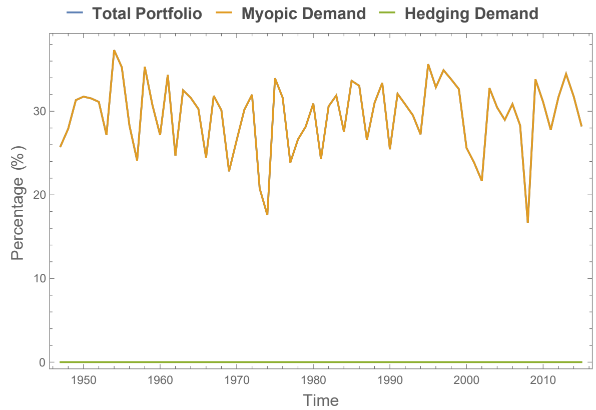


(e) Time–Series Consumption $(20, 0.75)$

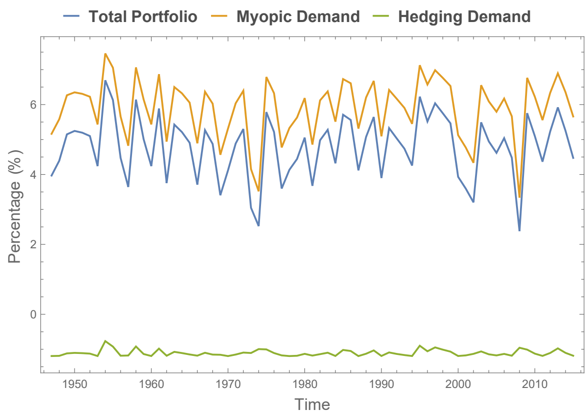
Figure 4: Implied Time–Series Consumption—Wealth Ratio with (γ, ψ). The annual estimated parameters are $\mu_H = 0.0757$, $\mu_L = r$, $\lambda_H = 0.3155$, $\lambda_L = 0.6667$, $r = 0.0067$, $\sigma = 0.1586$, and $\beta = 0.06$.



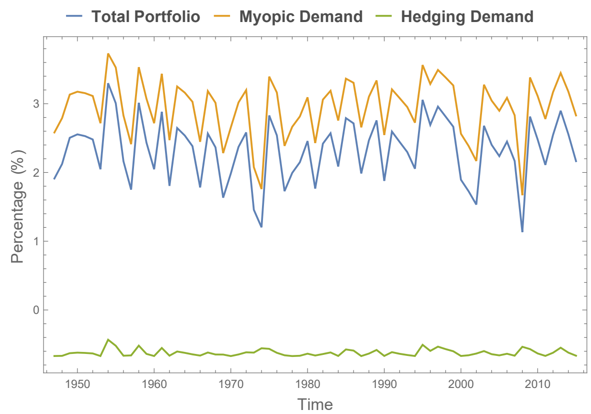
(a) Time–Series Portfolio $(0.75, 0.75)$



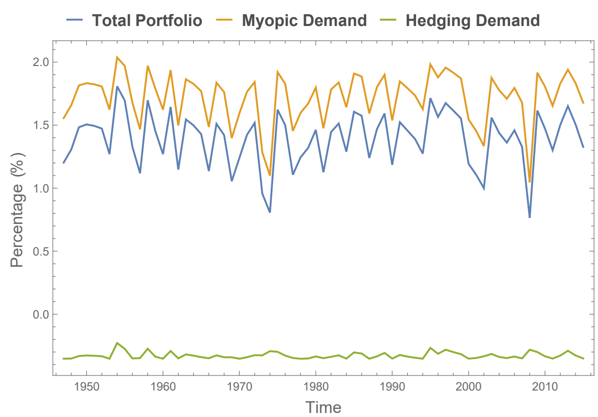
(b) Time–Series Portfolio $(1, 0.75)$



(c) Time–Series Portfolio $(5, 0.75)$



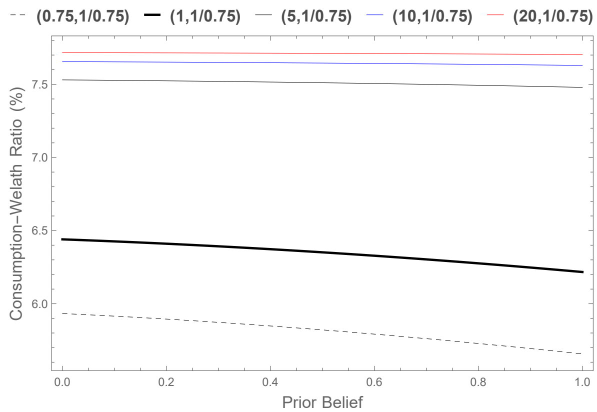
(d) Time–Series Portfolio $(10, 0.75)$



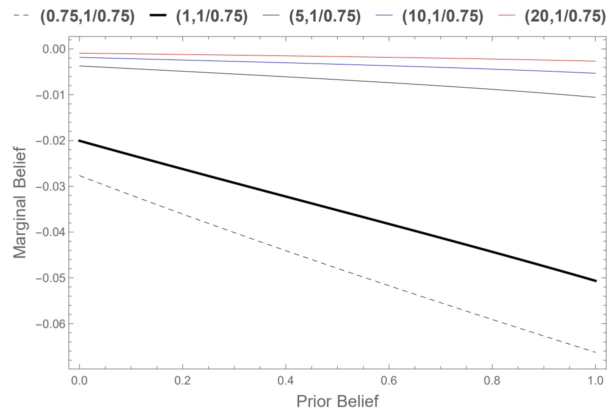
(e) Time–Series Portfolio $(20, 0.75)$

Figure 5: Implied Time–Series Portfolio Proportion with (γ, ψ). The annual estimated parameters are $\mu_H = 0.0757$, $\mu_L = r$, $\lambda_H = 0.3155$, $\lambda_L = 0.6667$, $r = 0.0067$, $\sigma = 0.1586$, and $\beta = 0.06$.

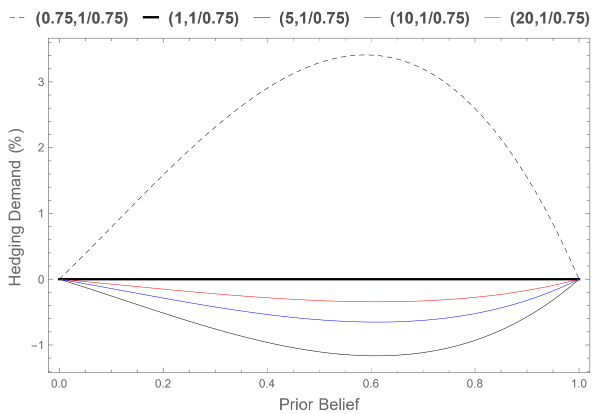
Section 3.2: An Approximate Solution with $EIS = 1/0.75 > 1$



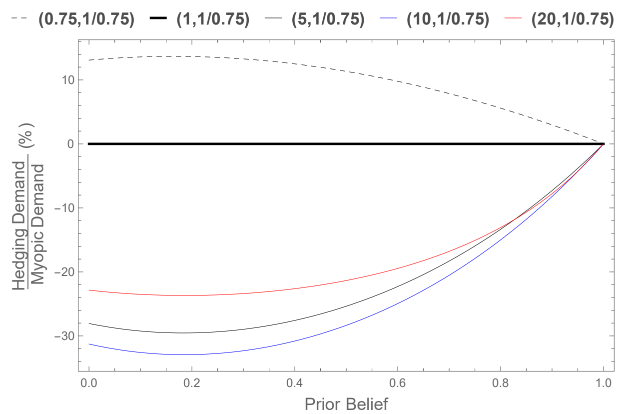
(a) Consumption–Wealth Ratio



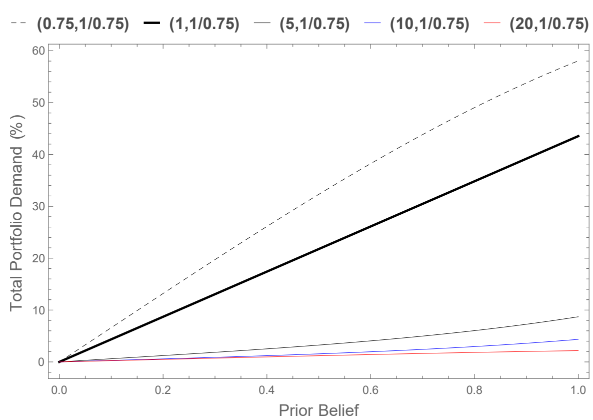
(b) Marginal Belief h_p



(c) Hedging Demand

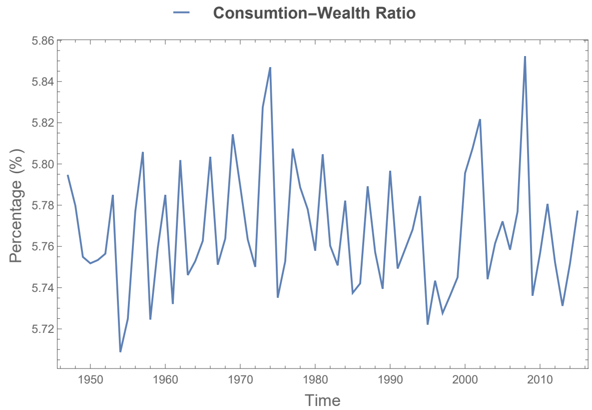


(d) Ratio of Hedging Demand over Myopic Demand

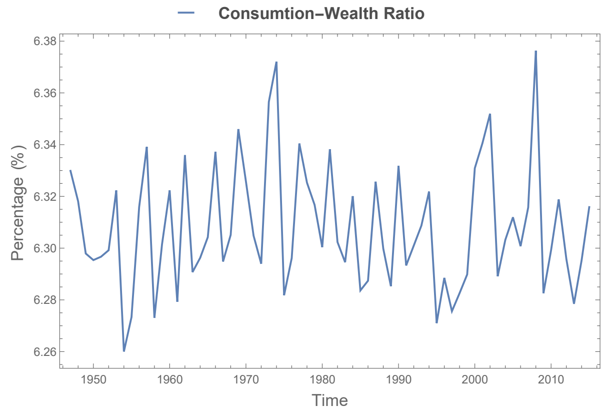


(e) Optimal Portfolio

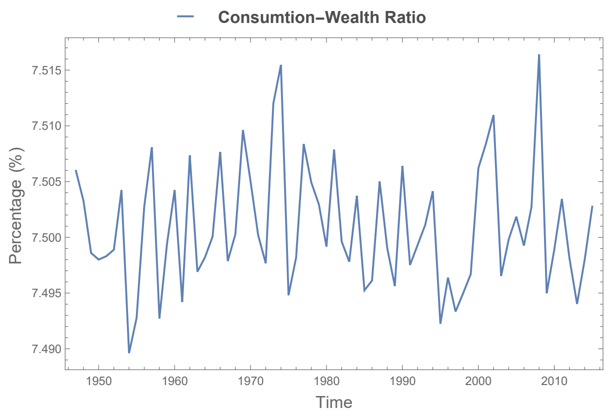
Figure 6: Optimal Consumption and Portfolio Policies with (γ, ψ). The annual estimated parameters are $\mu_H = 0.0757$, $\mu_L = r$, $\lambda_H = 0.3155$, $\lambda_L = 0.6667$, $r = 0.0067$, $\sigma = 0.1586$, and $\beta = 0.06$.



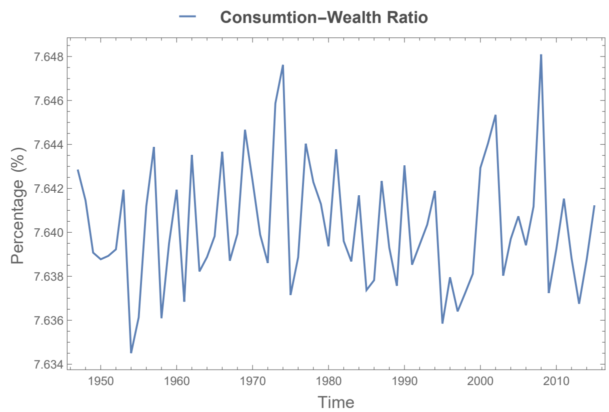
(a) Time–Series Consumption $(0.75, 1/0.75)$



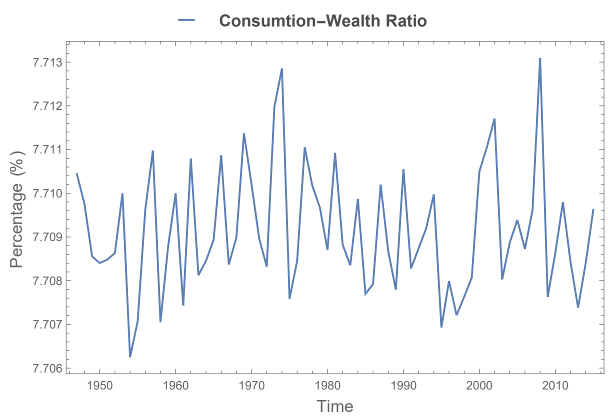
(b) Time–Series Consumption $(1, 1/0.75)$



(c) Time–Series Consumption $(5, 1/0.75)$

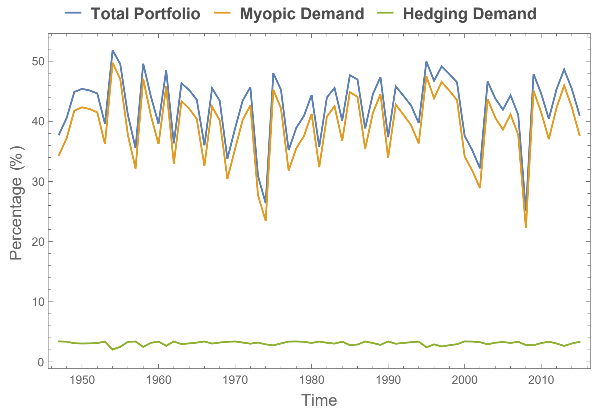


(d) Time–Series Consumption $(10, 1/0.75)$

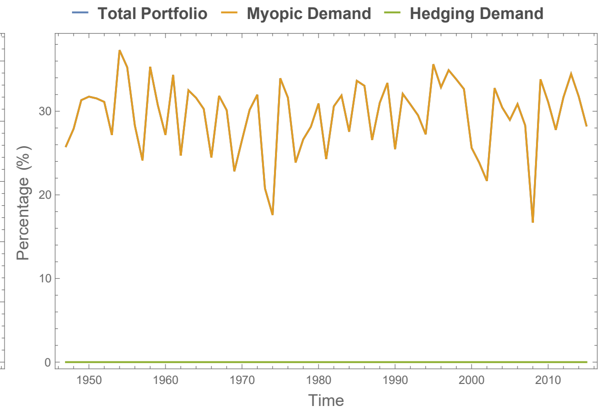


(e) Time–Series Consumption $(20, 1/0.75)$

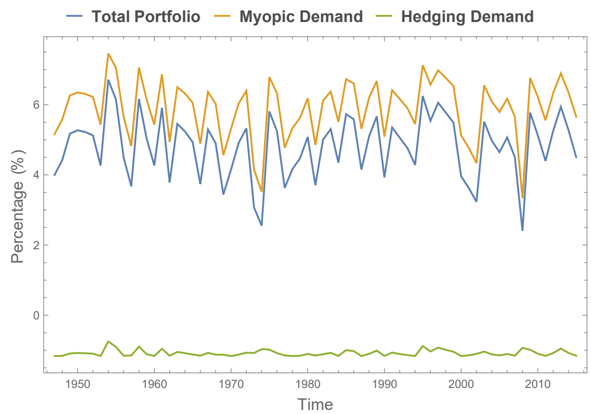
Figure 7: Implied Time-Series Consumption–Wealth Ratio with (γ, ψ). The annual estimated parameters are $\mu_H = 0.0757$, $\mu_L = r$, $\lambda_H = 0.3155$, $\lambda_L = 0.6667$, $r = 0.0067$, $\sigma = 0.1586$, and $\beta = 0.06$.



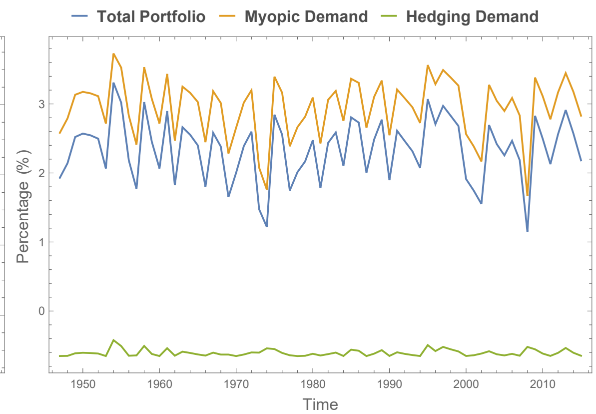
(a) Time–Series Portfolio $(0.75, 1/0.75)$



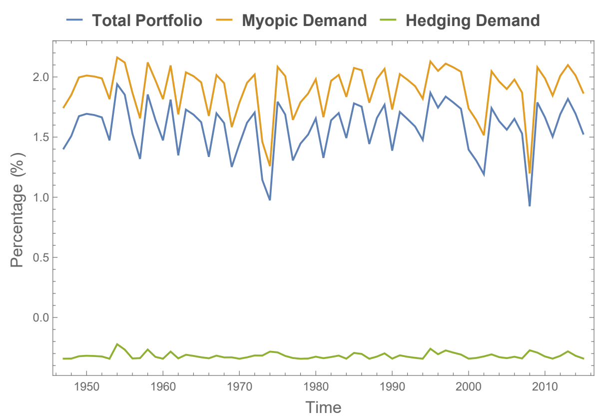
(b) Time–Series Portfolio $(1, 1/0.75)$



(c) Time–Series Portfolio $(5, 1/0.75)$



(d) Time–Series Portfolio $(10, 1/0.75)$



(e) Time–Series Portfolio $(20, 1/0.75)$

Figure 8: Implied Time–Series Portfolio Proportion with (γ, ψ). The annual estimated parameters are $\mu_H = 0.0757$, $\mu_L = r$, $\lambda_H = 0.3155$, $\lambda_L = 0.6667$, $r = 0.0067$, $\sigma = 0.1586$, and $\beta = 0.06$.