Two-Stage Startup Financing with Signaling under Ambiguity

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Abstract

We develop a model of two-stage startup financing with signaling under ambiguity. In our model, the nature determines the ability of technology entrepreneur and he strategically chooses a costly patent level as a signal to inform his ability to potential investors. Angels participate in investments for seed money in the first stage after observing the patent level and then venture capitals offer investments to the entrepreneur in the second stage based on angels' behavior. We provide three different financing models in view of the degree of ambiguity: (1) no ambiguity; (2) only investors face ambiguity; (3) all agents face ambiguity. In each signaling game between the entrepreneur and investors who may have ambiguous beliefs about the types of the entrepreneur, we find a unique perfect Bayesian equilibrium by imposing the Intuitive Criterion of Cho and Kreps (1987) and characterize the refined equilibria. In particular, angels ask the highest level of patents in order to ensure the ability of the entrepreneur when only investors have ambiguous information. We also find that the entrepreneur can obtain a higher utility under ambiguity than without it if his project is sufficiently overvalued. On the other hand, when investors have ambiguous information, the entrepreneur can be better off by resolving ambiguity if the project is sufficiently undervalued.

Keywords: startup financing; angel; venture capital; signaling; perfect Bayesian equilibrium; Intuitive Criterion

JEL Classification: G14, G24, D82

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1 Introduction

One of the most important issues for startup companies is to secure financing. Due to the absence of track records, it is essential for startups to inform the success probability of their projects to potential investors. Thus startups need to reveal reliable information about their ability to attract investors in early financing stages. For startups with technology, the number of patents filed can be a useful signal to access seed investors. As Graham et al. (2009) point out, technology startups tend to hold patents for competitive advantage, securing financing, and enhancing reputation. Analyzing Berkely Patent Survey, they find that holding more patents can make it easy for startups to be funded from external investors. Furthermore, Conti, Thursby, and Rothaermel (2013) empirically show that, in startup financing, an increase of patents level raises of both the frequency and amount of investments from venture capitals (VCs). According to Conti, Thursby, and Thursby (2013), the number of patents are strategically chosen by entrepreneurs to attract new investors.

Another important issue for startup company is concerned with ambiguity, which is incomplete information about success probability. Since the project of a startup company is innovative and has few track records, it is plausible that the entrepreneur or investors may not have a single prior belief about the project's success probability. They are likely to have insufficient knowledge of the probability and thus face ambiguity. Indeed, Rigotti et al. (2008) point out that technology startups often have ambiguous information about projects return. Recently, Kim and Wagman (2016) propose a theoretic startup financing model in which the entrepreneur and investors have ambiguous beliefs about the success probability.

Typically, there are two major types of investors who participate in startup financing markets: business angels and venture capitals. Angels are wealthy individuals who usually provide seed capitals to startup companies.[1] VCs provide professional support services as well as capitals to entrepreneurs usually at a later stage.[2] Conti, Thursby, and Thursby (2013) show that private investors tend to participate more in a seed financing stage while VCs tend to invest more in later stages.[3] According to Wong et al. (2009), if VCs participate in the second-stage financing, angels generally do not. On the other hand, using a theoretical model, Kim and Wagman (2016) show that an entrepreneur prefers to be financed by angels in the first-stage if the entrepreneur's ability is unknown to all the agents.

There are growing literature considering a signaling game between an entrepreneur and external investors in early-stage financing. Conti, Thursby, and Thursby (2013) propose a single-stage financing model, in which a technology entrepreneur strategically chooses the number of patents to attract potential investors in a seed investment market. Conti, Thursby, and Rothaermel (2013)

[1]See Fenn et al. (1997) and Wong et al. (2009) for the roles of angels in startup financing.

[2]See Denis (2004), Kaplan and Stromberg (2000), and Hellman and Puri (2002) for the roles of VCs in startup financing.

[3]See also Elitzur and Gavious (2003) and Wong et al. (2009).

also develop a single-stage financing model, in which the amount of money invested by acquaintances is used as a signal in addition to patent level. In both articles, an entrepreneur sends signals to access seed investors and the future value of a startup depends on signals chosen by the entrepreneur as in Spence (1974).[4] Two-stage financing model with signaling is proposed by Kim and Wagman (2016). In their model, the entrepreneur chooses either an offer from angels or that from VCs in the first stage, and the choice by the entrepreneur becomes a signal to the second-stage investors. In Kim and Wagman (2016), signaling does not affect the entrepreneurial value as in Spence (1973).

The purpose of this paper is to analyze how a startup's patent signaling affects a two-stage financing under ambiguity. To do this, we provide models in which an entrepreneur strategically chooses costly patent level as a signaling device to inform his ability to potential investors. Following Spence (1973), we assume that future project value is not affected by the patent level but affected by the inborn ability of the entrepreneur. We consider a two-stage financing model. In the first stage, angels participate in a seed investment to initiate the entrepreneur's project based on the entrepreneur's patent choice. In the second stage, after observing angel's behavior, VCs offer follow-up investment, which is necessary to continue the project. Angels play roles as both a receiver in the signaling game and a Stackelberg leader in the investment market. When making investment choices, angels optimally respond to the observed a signal of patent level as well as taking into account the best response of VCs. On the other hand, VCs indirectly observe the patent signal from the behavior of angels.

Unlike the literature of signaling game, we consider the cases where agents may have ambiguous beliefs about the entrepreneur's type (ability). To model ambiguity attitude, we adopt the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005), who represent preferences by the expected distortion of the expected utility, and consider its special case where agents are risk-neutral and ambiguity-neutral. To examine the effects of ambiguity, we provide three different financing models in view of the degrees of ambiguity: (1) no ambiguity; (2) only investors face ambiguity; (3) all agents face ambiguity. In the first model, the entrepreneur has the exact information about his success probability and investors know the success probability of each type. We consider the first model as a benchmark. In the second model, the entrepreneur still has the exact information about the success probability, but investors have ambiguous beliefs about the distribution of each type's success probability. The third model consider the case where both the entrepreneur and investors face ambiguity of beliefs about the success probability.

We derive perfect Bayesian equilibria in each two-stage signaling games of three different financing models. We refine them into a unique equilibrium by imposing the Intuitive Criterion of Cho and Kreps (1987) for each model. In refined equilibria, it is shown that, to signal his abil-

[4]In Spence (1974), productivity of a worker depends on his level of education while it is independent of the education level in Spence (1973).

ity, the entrepreneur should spend more money on filing patents in the model where only investors face ambiguity than other models. It is because, to avoid failing to separate the entrepreneur's type, somewhat strongly incentive compatibility constraints are considered by angels who face ambiguity with knowing that the entrepreneur resolves it. Thus resolving ambiguity has no effect on saving entrepreneur's patents costs. We also find that the entrepreneur can be better off when ambiguity is present than otherwise if his project is sufficiently overvalued by investors. On the other hand, when investors have ambiguous information, the entrepreneur can increase his utility by resolving ambiguity if the project is sufficiently undervalued.

Our model is related to Elitzur and Gavious (2003) and Kim and Wagman (2016) which study two-stage financing models for startups. Similar to our model, in Elitzur and Gavious (2003), an angel invests in the first stage and a venture capital does in the second stage. However, they focus on the role of the angel as an advisor of the entrepreneur. In their model, the angel initially determines the shares of all agents including her own, and then the entrepreneur and the venture capital only choose effort level and the amount of investment, respectively. Another major difference of their model from ours is that the angel moves before the entrepreneur. They assume that angel approaches the entrepreneur and makes investment offer, and then the entrepreneur chooses effort level based on the offer. In our model, on the contrary, the entrepreneur first reveals his own patent level to attract angels, who then make investment offer after observing the patent level.

On the other hand, Kim and Wagman (2016) propose a model in which only VCs have an option of reinvestment in the second stage, but it is impossible for angels to participate in the second stage after providing finance in the first stage. Although their model considers a signaling game between an entrepreneur and investors as in our model, they do not consider the effort taken by the entrepreneur to access seed capital. They assume that the entrepreneur can choose seed investors without signaling and the choice signals his ability to later stage investors. Consequently, this study complements Elitzur and Gavious (2003) and Kim and Wagman (2016) in that we consider a signal to attract seed investors.

Our model is also related to the single-stage financing models of Conti, Thursby, and Rothaermel (2013) and Conti, Thursby, and Thursby (2013) which adopt patent level as a signal in order to attract seed investors. Unlike their models, however, we assume that the project value is independent of the entrepreneur's signal so that the signal is unproductive. We consider the case where the value of the project strongly depends on the inborn ability of the entrepreneur. This assumption allows us to develop a more simplified two-stage financing model and to derive closed solutions, so that we can focus on characterizing equilibria.

The rest of the paper is organized as follows. In Section 2, we introduce three models classified by the degree of ambiguity where we consider two-stage entrepreneurial financing with signaling. We derive perfect Bayesian equilibria of the signaling games for each model in Section 3 and refine them in view of the Intuitive Criterion of Cho and Kreps (1987) in Section 4. We characterize our refined equilibria by comparative statics in Section 5. Concluding remarks are given in Section 6. All the proofs are relegated to Appendix.

2 Model

There are three types of risk-neutral and ambiguity-neutral agents: a technology entrepreneur, angels, and VCs.[5] The sequence of event consists of four periods $\tau = 0, 1, 2, 3$. At $\tau = 0$, the nature determines success probability s of a project which the entrepreneur will launch. The entrepreneur's risky project generates random asset value R, which is defined on $\Omega \equiv \{\text{success}, \text{failure}\}$ and has a binomial distribution such that

$$R(\omega) = \begin{cases} A & \text{if } \omega = \text{success}, \\ 0 & \text{if } \omega = \text{failure}, \end{cases}$$

with success probability is s. Thus the expected project value is $\mathbb{E}_s[R] = sA$.

Let s_H and s_L denote the true success probabilities of high type and low type of the entrepreneur, respectively. Suppose that the project is the most innovative and thus has no track record. Then the entrepreneur and investors may not learn the exact success probability at $\tau = 0$. In other words, they may be faced with ambiguity about success probability s. Without ambiguity, the agents know the exact value of s. Facing ambiguity, on the other hand, they suppose that s has a distribution ν, which we assume is a uniform distribution on $[0, 1]$. Let s^* denote a threshold probability where $s_L < s^* \leq s_H$. The entrepreneur with high ability has success probability $s_H \in [s^*, 1] \equiv I_H$ and with low ability has success probability $s_L \in (0, s^*) \equiv I_L$. We call the type who belongs to I_H the *high type* and who belongs to I_L the *low type*. Note that threshold probability s^* can be interpreted as markets' evaluation about success probability since a high s^* means that the project is expected to yield a high asset value.

In the environment of startup financing, the project is typically innovative and has no track record. Then the entrepreneur or investors may not learn the exact success probability at $\tau = 0$. This means that ambiguity about the success probability plays an important role in decision making for startup financing. The uncertainty of probability distribution can be modeled by employing the maximin expected utility model (Gilboa and Schmeidler, 1989), the multiplier model (Hansen and Sargent, 2001), the smooth ambiguity model (Klibanoff, Marinacci, and Mukerji, 2005), or the variational preference model (Maccheroni, Marinacci, and Rustichini, 2006). In this paper, we adopt the smooth ambiguity model and consider ambiguity-neutrality as a special case.[6]

[5] Ambiguity neutrality applies only to the agents with ambiguous beliefs.

[6] In Klibanoff et al. (2005), a smooth ambiguity preference $\succcurlyeq$ is represented by the function V:

$$V(f) = \int_{\Delta} \phi \left(\int_S u(f(\omega)) d\pi(\omega) \right) d\mu(\pi) \equiv \mathbb{E}_\mu \left[\phi \left(\mathbb{E}_\pi [u(f)] \right) \right],$$

To reflect ambiguous beliefs of the entrepreneur or investors, we introduce two possibly different type spaces for the entrepreneur and investors, denoted by T and $\hat{T}$, respectively. To analyze the effects of ambiguity on startup financing, we consider three cases. First, as a benchmark, we consider a case where the entrepreneur know his exact type and investors are informed about possible types, in which $T = \hat{T} = \{s_H, s_L\}$. In this case, investors have prior beliefs μ and π about the types of the entrepreneur, respectively such that $\mu(s_H) = \pi(s_H) = q \in (0,1)$. In the second case, only the entrepreneur learns his exact type while investors have multiple beliefs about the possible types. That is, investors face ambiguity about the success probability of the project. We assume that investors only know whether $s \in I_H$ or $s \in I_L$. Thus $T = \{s_H, s_L\}$ and $\hat{T} = \{I_H, I_L\}$. Third, we consider a case where all the agents do not learn possible types and have ambiguous information about s. This implies that $T = \hat{T} = \{I_H, I_L\}$. In the second and third case, investors have prior beliefs μ and π about the types of the entrepreneur, respectively such that $\mu(H) = \pi(H) = q \in (0,1)$.

In order to attract investors, the entrepreneur strategically choose patent level $\psi \in [0, \infty)$ after learning his type. According to Conti, Thursby, and Rothaermel (2013), the chosen patent level is related to commitment of the entrepreneur in that the patent level increases in the expense of the entrepreneur's own money. We consider that the high type can acquire patents more efficiently than the low type. The entrepreneur's cost function of acquiring patent is given by

$$c(s,\psi) = \begin{cases} c_H\psi & \text{if } s \in I_H, \\ c_L\psi & \text{if } s \in I_L, \end{cases}$$

where $c_L > c_H$.

All investors are divided into two groups: angels and VCs. Angels are wealthy individuals who provide seed capital to the entrepreneur in the first-stage. Under Bertrand competition, angels offer equity shares as their portions and the entrepreneur picks an angel who offers the lowest share. VCs are specialized investment organizations, which usually participate in a later stage after the project is successfully initiated. As in the seed investment market, VCs offer equity shares to have and the entrepreneur chooses a VC who offers the lowest share. The follow-up investment market is also under Bertrand competition. Thus the expected profits of both angels and VCs should be zero. Since we assume Bertrand competition in both investment markets, we consider that all investors are represented by a single angel and VC.

where u is a continuous strictly increasing utility function, ϕ is a continuous strictly increasing distortion function, Δ is a set of probability measure π's, and μ is a countably additive probability measure over Δ. In particular, preference $\succcurlyeq$ is *ambiguity neutral* if ϕ is linear (see Corollary of Klibanoff et al. (2005)). Moreover, in the case of risk-neutrality and ambiguity-neutrality as in our model, both u and ϕ are linear functions. In our context, we represent the preferences of an agent over R by

$$V(R) = \int_{[0,1]} [\mathbb{E}_s[R]]ds = \mathbb{E}_\nu[\mathbb{E}_s[R]],$$

where ν is the uniform probability measure of s on $[0,1]$.

2.1 First-Stage Financing

To launch the project, the entrepreneur needs seed investment K_1 at $\tau = 1$. In the fist-stage, the angel, who has observed entrepreneur's patent level ψ approaches the entrepreneur and offers investment K_1 for her share $\beta \in (0, 1)$. The angel does not directly observe whether entrepreneur's type is high or low, but they update her belief about the type of the entrepreneur after observing ψ. Note that the angel plays a role as both a receiver of the signaling game with the entrepreneur and a Stackelberg leader who understands the optimal response function of the VC in the second-stage financing. Thus, when making decision, the angel considers observed patent level of the entrepreneur and possible investment choices of the VC at the same time.

2.2 Second-Stage Financing

To continue the project, follow-up investment K_2 is required at $\tau = 2$. Without the follow-up investment, the project must be shut down and revenue is not generated, i.e., $R = 0$. After observing the behavior of the angel, the VC offers investment K_2 for share $\gamma \in (0, 1)$. Note that the VC cannot also directly observe the type of the entrepreneur. By updating beliefs about the type after observing β, they move as a Stackelberg follower.

As a result of the second-stage financing, the angel's share β at $\tau = 1$ is diluted and her final share becomes $\beta(1 - \gamma)$. The entrepreneur retains the share net of investors' portion. Therefore, the final share of the entrepreneur is given by

$$\theta \equiv 1 - [\beta(1 - \gamma) + \gamma] .$$

At $\tau = 3$, the value R of the project is realized, and if the project succeeds, the entrepreneur, the angel and the VC are paid proportional to their equity shares. We illustrate the sequence of event in Figure 1 below.

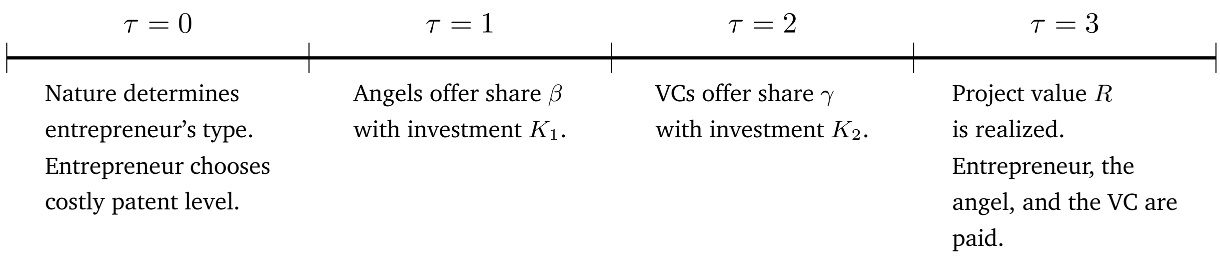

Figure 1: Sequence of events

2.3 Payoffs of Agents

Now we define the (expected) utility functions of agents. Let $\mathbb{E}_\eta[\cdot]$ denote the expected value under probability measure η. Since all the agents are risk-neutral, their utilities are defined by retained cash-out shares. Thus the utility function of the entrepreneur is given by

$$u(t,\psi,\beta,\gamma) = \theta R - c_t\psi.$$

Since the entrepreneur chooses patent level ψ after learning his type, his expected utility is[7]

$$\begin{aligned} U(t,\psi,\beta,\gamma) &= \theta\mathbb{E}_\nu[\mathbb{E}_s[R]|t] - c_t\psi \\ &= \theta\mathbb{E}_\nu\left[\tilde{s}|\, t\right] A - c_t\psi \end{aligned}$$

where ν is the uniform probability measure of s on $[0,1]$. The angel's utility function is given by

$$v(\beta,\gamma) = (1-\gamma)\,\beta R - K_1.$$

She makes investment decision based on an observed patent level of the entrepreneur and the VC's best response. The interim expected utility of the angel is[8]

$$\begin{aligned} \tilde{V}(t,\psi,\beta,\gamma) &= (1-\gamma)\,\beta\mathbb{E}_\nu[\mathbb{E}_s[R]|t] - K_1 \\ &= (1-\gamma)\,\beta\mathbb{E}_\nu\left[\tilde{s}\,|t\right] A - K_1. \end{aligned}$$

Thus the expected utility of the angel is given by

$$V(\psi,\beta,\gamma) = \sum_{t\in T}\mu(t|\psi)\tilde{V}(t,\psi,\beta,\gamma)\,.$$

Finally, the VC's utility function is given by

$$w(\gamma) = \gamma R - K_2.$$

Its interim expected utility is

$$\begin{aligned} \tilde{W}(t,\beta,\gamma) &= \gamma\mathbb{E}_\nu[\mathbb{E}_s[R]|t] - K_2 \\ &= \gamma\mathbb{E}_\nu\left[\tilde{s}\,|t\right] A - K_2. \end{aligned}$$

Thus the expected utility of the VC is

$$W(\beta,\gamma) = \sum_{t\in T}\pi(t|\beta)\tilde{W}(t,\beta,\gamma).$$

To ensure the participation of investors, we assume that

$$\min\left\{s_L A, \frac{1}{2}s^* A\right\} > K_1 + K_2 \tag{2.1}$$

[7] Facing ambiguity about the distribution of s, agents consider s as a random variable. Thus we use notation $\tilde{s}$ here.
[8] Here, "interim" means that the investors know the type of the entrepreneur.

3 Equilibrium

We adopt the solution concept of perfect Bayesian equilibrium (PBE). In this section, working backward, we derive two kinds of equilibria: separating equilibria and pooling equilibria. As mentioned in Section 2, agents may have multiple beliefs about the success probability of the project. We consider three cases. In the first case, all the agents have the exact information about the distribution of the success probability s. This case is treated as a benchmark. The second and third cases assume that some or all the agents face ambiguity. In the second case, only investors have multiple beliefs about the distribution of s, and in the third case, all agents do so.

3.1 Benchmark Model: No Ambiguity

In this subsection, the entrepreneur learns his type (success probability) and investors know possible types of the entrepreneur at $\tau = 0$.[9] The analysis is analogous to that in the job market signaling game of Spence (1973) except for the sequential investments.

3.1.1 Separating Equilibria

In the separating equilibria, the type of the entrepreneur is perfectly revealed. Suppose that, in the fist stage, the angel offers β_H^0 if they believe that the entrepreneur's type is s_H and offer β_L^0 if they believe that his type is s_L. In the second stage, VC who observes the angel's offer β_H^0 believe that the entrepreneur's type is s_H and who observe β_L^0 believe that his type is s_L. Since the VC's expected utility should be zero, i.e., $W(\beta, \gamma) = 0$, we have

$$W\left(\beta_H^0, \gamma_H^0\right) = \gamma_H s_H A - K_2 = 0, \quad \text{and} \quad W\left(\beta_L^0, \gamma_L^0\right) = \gamma_L s_L A - K_2 = 0, \tag{3.1}$$

which implies

$$\gamma_H^0 = \frac{K_2}{s_H A} \quad \text{and} \quad \gamma_L^0 = \frac{K_2}{s_L A}. \tag{3.2}$$

Note that $0 < \gamma_H^0 < \gamma_L^0 < 1$ by (2.1) and (3.2). The investment of the VC in the second stage dilutes of the angel's share. If the entrepreneur's type is s_H, her share becomes $\beta_H^0\left(1-\gamma_H^0\right)$ and if his type is s_L, her share becomes $\beta_L^0\left(1-\gamma_L^0\right)$. Suppose that the angel believes that the type is s_H if they observe patent level ψ_H^0 and believe that the type is s_L if they observe patent level ψ_L^0. Having zero expected utility, i.e., $V(\psi, \beta, \gamma) = 0$, the angel solves

$$\begin{aligned} V\left(\psi_H^0, \beta_H^0, \gamma_H^0\right) &= \beta_H^0\left(1-\gamma_H^0\right) s_H A - K_1 = 0 \\ V\left(\psi_L^0, \beta_L^0, \gamma_L^0\right) &= \beta_L^0\left(1-\gamma_L^0\right) s_L A - K_1 = 0, \end{aligned} \tag{3.3}$$

[9] The entrepreneur's type is identified with his success probability.

which implies

$$\beta_H^0 = \frac{K_1}{s_H A - K_2} \quad \text{and} \quad \beta_L^0 = \frac{K_1}{s_L A - K_2}. \tag{3.4}$$

From (2.1) and (3.4), we know that $0 < \beta_H^0 < \beta_L^0 < 1$. Angel's share after dilution for the high type and the low type are given by

$$\beta_H^0 \left(1 - \gamma_H^0\right) = \frac{K_1}{s_H A} \quad \text{and} \quad \beta_L^0 \left(1 - \gamma_L^0\right) = \frac{K_1}{s_L A}.$$

The share of the entrepreneur with type s_H and that of him with type s_L are given by, respectively,

$$\begin{aligned} \theta_H^0 &\equiv 1 - \left[\beta_H^0 \left(1 - \gamma_H^0\right) + \gamma_H^0\right] = 1 - \frac{K_1 + K_2}{s_H A}, \\ \theta_L^0 &\equiv 1 - \left[\beta_L^0 \left(1 - \gamma_L^0\right) + \gamma_L^0\right] = 1 - \frac{K_1 + K_2}{s_L A}. \end{aligned} \tag{3.5}$$

Note that $0 < \theta_L^0 < \theta_H^0 < 1$ by (2.1) and (3.5). From (3.5), the entrepreneur's expected utility is given by

$$U\left(t, \psi, \beta, \tilde{\gamma}(\beta)\right) = \begin{cases} \theta_H^0 s_H A - c_H \psi & \text{if } t = s_H, \\ \\ \theta_L^0 s_L A - c_L \psi & \text{if } t = s_L. \end{cases}$$

To separate the types of the entrepreneur, the angel considers incentive compatibility constraints such that

$$\theta_L^0 s_L A \geq \theta_H^0 s_L A - c_L \psi_H^0 \quad \text{and} \quad \theta_H^0 s_H A - c_H \psi_H^0 \geq \theta_L^0 s_H A. \tag{3.6}$$

Then we have the following proposition.

Proposition 3.1. *There are separating perfect Bayesian equilibria, where the patent levels of type s_L and type s_H entrepreneurs are given by*

$$\psi_L^0 = 0, \quad \textit{and} \quad \psi_H^0 \in \left[\psi^0, \frac{(s_H - s_L)(K_1 + K_2)}{s_L c_H}\right],$$

with

$$\psi^0 \equiv \frac{(s_H - s_L)(K_1 + K_2)}{s_H c_L},$$

the posterior belief and the offer of the angel are given by, respectively,

$$\mu(H|\psi) = \begin{cases} 0 & \textit{if } \psi < \psi_H^0, \\ 1 & \textit{if } \psi \geq \psi_H^0, \end{cases} \quad \textit{and} \quad \tilde{\beta}(\psi) = \begin{cases} \beta_L^0 & \textit{if } \psi < \psi_H^0, \\ \beta_H^0 & \textit{if } \psi \geq \psi_H^0, \end{cases} \tag{3.7}$$

and the posterior belief and the offer of the VC are given by, respectively,

$$\pi(H|\beta) = \begin{cases} 0 & \textit{if } \beta > \beta_H^0, \\ 1 & \textit{if } \beta \leq \beta_H^0, \end{cases} \quad \textit{and} \quad \tilde{\gamma}(\beta) = \begin{cases} \gamma_L^0 & \textit{if } \beta > \beta_H^0, \\ \gamma_H^0 & \textit{if } \beta \leq \beta_H^0. \end{cases} \tag{3.8}$$

3.1.2 Pooling Equilibria

Under pooling equilibria, the angel cannot identify the entrepreneur's type. Suppose that the angel offers share β_P^0 in the first-stage financing. Then neither can the VC extract information about the entrepreneur's type from the share offered by the angel. Let γ_P^0 denote the VC's investment offer after observing β_P^0. Since $W(\beta, \gamma) = 0$, we have

$$W\left(\beta_P^0, \gamma_P^0\right) = \left[qs_H + (1-q)s_L\right]\gamma_P^0 A - K_2 = 0,$$

which implies

$$\gamma_P^0 = \frac{K_2}{\left[\left(s_H - s_L\right)q + s_L\right]A}. \tag{3.9}$$

Since $V(\psi, \beta; \gamma) = 0$, the angel solves

$$V\left(\psi_P^0, \beta_P^0, \gamma_P^0\right) = \left[qs_H + (1-q)s_L\right]\left(1 - \gamma_P^0\right)\beta_P^0 A - K_1 = 0,$$

which implies

$$\beta_P^0 = \frac{K_1}{\left[(1-q)s_L + qs_H\right]A - K_2}. \tag{3.10}$$

Then the share of the entrepreneur is given by

$$\theta_P^0 \equiv 1 - \left[\left(1 - \gamma_P^0\right)\beta_P^0 + \gamma_P^0\right] = 1 - \frac{K_1 + K_2}{\left[(1-q)s_L + qs_H\right]A}. \tag{3.11}$$

Note that $\theta_P^0 \neq q\theta_H^0 + (1-q)\theta_L^0$.[10] Clearly, γ_P^0, β_P^0, and θ_P^0 belong to $(0,1)$ from (2.1). From (3.11), the entrepreneur's expected utility is given by

$$U\left(t, \psi, \beta, \tilde{\gamma}(\beta)\right) = \begin{cases} \theta_P^0 s_H A - c_H \psi & \text{if } t = s_H, \\ \\ \theta_P^0 s_L A - c_L \psi & \text{if } t = s_L. \end{cases}$$

To pool the types of the entrepreneur, the angel considers incentive compatibility constraints such that

$$\theta_P^0 s_L A - c_P \psi_P^0 \geq \theta_L^0 s_L A \quad \text{and} \quad \theta_P^0 s_H A - c_H \psi_P^0 \geq \theta_L^0 s_H A. \tag{3.12}$$

Proposition 3.2. *There are pooling perfect Bayesian equilibria, where the patent level for the entrepreneur is given by*

$$\psi_P \in \left[0, \frac{q\left(s_H - s_L\right)\left(K_1 + K_2\right)}{c_L\left[(1-q)s_L + qs_H\right]}\right],$$

the posterior belief and the offer for the angel are given by, respectively,

$$\mu(H|\psi) = \begin{cases} 0 & \text{if } \psi \neq \psi_P \\ q & \text{if } \psi = \psi_P \end{cases} \quad \text{and} \quad \tilde{\beta}(\psi) = \begin{cases} \beta_L^0 & \text{if } \psi \neq \psi_P, \\ \beta_P^0 & \text{if } \psi = \psi_P, \end{cases} \tag{3.13}$$

[10] This is different result from that of general job market signaling game.

and the posterior belief and the offer for the VC are given by, respectively,

$$\pi(H|\beta) = \begin{cases} 0 & \text{if } \beta \neq \beta_P \\ q & \text{if } \beta = \beta_P \end{cases} \quad \textit{and} \quad \tilde{\gamma}(\beta) = \begin{cases} \gamma_L^0 & \text{if } \beta \neq \beta_P, \\ \gamma_P^0 & \text{if } \beta = \beta_P. \end{cases} \tag{3.14}$$

3.2 Model I: Investors with Ambiguity

In this subsection, we assume that only the entrepreneur learns his success probability s at $\tau = 0$ while investors are uninformed about possible types. Angels who observe the patent level of the entrepreneur only know whether $s \in I_H$ or $s \in I_L$. Thus investors believe that the high type succeeds in the project with probability $(1+s^*)/2$ and the low type succeeds in the project with probability $s^*/2$. Generally, the type of a sender is perfectly revealed in a perfect Bayesian separating equilibrium of a signaling game. However, we consider that, in the presence of multiple beliefs, the angel cannot perfectly evaluate incentive compatibility conditions of the entrepreneur and offer a extremely conservative equity share contract, which will not fail equilibria.

3.2.1 Separating Equilibria

Suppose that, in the first-stage, the angel offers β_H if they believe that $s \in I_H$ and offer β_L if they believe that $s \in I_L$. We assume that if the angel offers β_H, the VC believes that the entrepreneur is high type and offer γ_H, and if the angel offers β_L, the VC believes that he is low type and offer γ_L. Due to zero expected utility condition i.e., $W(\beta, \gamma) = 0$, we have

$$W(\beta_H, \gamma_H) = \left(\frac{1+s^*}{2}\right) A\gamma_H - K_2 = 0, \quad \text{and} \quad W(\beta_L, \gamma_L) = \frac{s^*}{2} A\gamma_L - K_2 = 0, \tag{3.15}$$

which implies

$$\gamma_H = \frac{2K_2}{(1+s^*)A} \quad \text{and} \quad \gamma_L = \frac{2K_2}{s^* A}. \tag{3.16}$$

Clearly, $0 < \gamma_H < \gamma_L < 1$ by (2.1) and (3.16).

As in the benchmark model, the investment by the VC in the second-stage induces dilution of the participating the angel's share. If $s \in I_H$, her share becomes $\beta_H(1-\gamma_H)$ and if $s \in I_L$, her share becomes $\beta_L(1-\gamma_L)$. Suppose that the angel believes that $s \in I_H$ if $\psi = \psi_H$ and believe that $s \in I_L$ if $\psi = \psi_L$. Since $V(\psi, \beta, \gamma) = 0$, the angel solves

$$\begin{aligned} V(\psi_H, \beta_H, \gamma_H) &= \left(\frac{1+s^*}{2}\right) \beta_H (1-\gamma_H) A - K_1 = 0, \\ V(\psi_L, \beta_L, \gamma_L) &= \frac{s^*}{2} \beta_L (1-\gamma_L) A - K_1 = 0, \end{aligned} \tag{3.17}$$

which implies that

$$\beta_H = \frac{2K_1}{(1+s^*)\,A - 2K_2} \quad \text{and} \quad \beta_L = \frac{2K_1}{s^*A - 2K_2}. \tag{3.18}$$

Note that $0 < \beta_H < \beta_L < 1$ by (2.1) and (3.18). Angels' share after dilution for the high type and the low type are given by, respectively,

$$\beta_H\,(1-\gamma_H) = \frac{2K_1}{(1+s^*)\,A} \quad \text{and} \quad \beta_L\,(1-\gamma_L) = \frac{2K_1}{s^*A}. \tag{3.19}$$

From (3.16) and (3.18), we find the entrepreneur's share. Since the entrepreneur retains share net of investors', the equity shares of the high type and the low type are given by, respectively,

$$\begin{aligned}\theta_H &\equiv 1 - [\beta_H\,(1-\gamma_H) + \gamma_H] = 1 - \frac{2\,(K_1+K_2)}{(1+s^*)\,A},\\ \theta_L &\equiv 1 - [\beta_L\,(1-\gamma_L) + \gamma_L] = 1 - \frac{2\,(K_1+K_2)}{s^*A}.\end{aligned} \tag{3.20}$$

From (2.1) and (3.20), we know that $0 < \theta_L < \theta_H < 1$. The entrepreneur can retain a higher share when the angel believes that he is high type than when they believe that he is low one. From (3.20), the entrepreneur's expected utility is given by

$$U\,(t,\psi,\beta,\tilde{\gamma}(\beta)) = \begin{cases} \theta_H s_H A - c_H\psi & \text{if } t = s_H, \\ \\ \theta_L s_L A - c_L\psi & \text{if } t = s_L. \end{cases}$$

To ensure separating the types, the angel considers conservative incentive compatibility for the entrepreneur. Indeed, the following incentive compatibility constraints should hold: for all $\hat{s}_L \in L = (0, s^*)$

$$\theta_L \hat{s}_L A \geq \theta_H \hat{s}_L A - c_L \psi_H$$

and for all $\hat{s}_H \in H = [s^*, 1]$

$$\theta_H \hat{s}_H A - c_H \psi'_H \geq \theta_L \hat{s}_H A$$

which implies

$$\psi_H \in \left[\frac{(\theta_H - \theta_L)\,s^*A}{c_L}, \frac{(\theta_H - \theta_L)\,s^*A}{c_H}\right]. \tag{3.21}$$

Proposition 3.3. *There are separating perfect Bayesian equilibria, where the patent levels of type s_L and type s_H entrepreneurs are given by*

$$\psi_L = 0, \quad \textit{and} \quad \psi_H \in \left[\psi^*, \frac{2\,(K_1+K_2)}{(1+s^*)\,c_H}\right],$$

where

$$\psi^* \equiv \frac{2\,(K_1+K_2)}{(1+s^*)\,c_L},$$

the posterior belief and the offer of the angel are given by, respectively,

$$\mu(H|\psi) = \begin{cases} 0 & \textit{if } \psi < \psi_H, \\ 1 & \textit{if } \psi \geq \psi_H, \end{cases} \quad \textit{and} \quad \tilde{\beta}(\psi) = \begin{cases} \beta_L & \textit{if } \psi < \psi_H, \\ \beta_H & \textit{if } \psi \geq \psi_H, \end{cases} \tag{3.22}$$

and the posterior belief and the offer of the VC are given by, respectively,

$$\pi(H|\beta) = \begin{cases} 0 & \textit{if } \beta > \beta_H, \\ 1 & \textit{if } \beta \leq \beta_H, \end{cases} \quad \textit{and} \quad \tilde{\gamma}(\beta) = \begin{cases} \gamma_L & \textit{if } \beta > \beta_H, \\ \gamma_H & \textit{if } \beta \leq \beta_H. \end{cases} \tag{3.23}$$

3.2.2 Pooling Equilibria

Suppose that the angel offers share β_P after observing patent level ψ_P in the first-stage financing. Let γ_P denote the VC's share offered after observing β_P: $\tilde{\gamma}(\beta_P) = \gamma_P$. Since $W(\beta, \gamma) = 0$, we have

$$W(\beta_P, \gamma_P) = \left[q\left(\frac{1+s^*}{2}\right) + (1-q)\frac{s^*}{2}\right]\gamma_P A - K_2 = 0,$$

which implies

$$\gamma_P = \frac{2K_2}{(q+s^*)A}. \tag{3.24}$$

Angels solve

$$V(\psi_P, \beta_P, \gamma_P) = \left[q\left(\frac{1+s^*}{2}\right) + (1-q)\frac{s^*}{2}\right]\beta_P(1-\gamma_P)A - K_1 = 0$$

and we have

$$\beta_P = \frac{2K_1}{(q+s^*)A - 2K_2}. \tag{3.25}$$

The share of the entrepreneur is given by

$$\theta_P \equiv 1 - [\beta_P(1-\gamma_P) + \gamma_P] = 1 - \frac{2(K_1+K_2)}{(q+s^*)A}. \tag{3.26}$$

Note that $\theta_P \neq q\theta_H + (1-q)\theta_L$. From (2.1), we know that γ_P, β_P, and θ_P belong to $(0,1)$.

From (3.26), the entrepreneur's expected utility is given by

$$U(t, \psi, \beta, \tilde{\gamma}(\beta)) = \begin{cases} \theta_P s_H A - c_H \psi & \text{if } t = s_H, \\ \\ \theta_P s_L A - c_L \psi & \text{if } t = s_L. \end{cases}$$

As in the separating equilibria, to avoid failure in pooling the types, the angel considers conservative incentive compatibility for the entrepreneur. Thus, in pooling equilibria, the following incentive compatibility constraints should hold: for all $\hat{s}_L \in I_L$ and for all $\hat{s}_H \in I_H$,

$$\theta_P \hat{s}_L A - c_L \psi_P \geq \theta_L \hat{s}_L A \quad \text{and} \quad \theta_P \hat{s}_H A - c_H \psi_P \geq \theta_L \hat{s}_H A,$$

which implies $\psi_P = 0$.

Proposition 3.4. *There is a pooling perfect Bayesian equilibrium, where the patent level for the entrepreneur is given by $\psi_P = 0$, the posterior belief and the offer for the angel are given by, respectively,*

$$\mu(H|\psi) = \begin{cases} 0 & \text{if } \psi \neq \psi_P \\ q & \text{if } \psi = \psi_P \end{cases} \quad \text{and} \quad \tilde{\beta}(\psi) = \begin{cases} \beta_L & \text{if } \psi \neq \psi_P, \\ \beta_P & \text{if } \psi = \psi_P, \end{cases} \tag{3.27}$$

and the posterior belief and the offer for the VC are given by, respectively,

$$\pi(H|\beta) = \begin{cases} 0 & \text{if } \beta \neq \beta_P \\ q & \text{if } \beta = \beta_P \end{cases} \quad \text{and} \quad \tilde{\gamma}(\beta) = \begin{cases} \gamma_L & \text{if } \beta \neq \beta_P, \\ \gamma_P & \text{if } \beta = \beta_P. \end{cases} \tag{3.28}$$

3.3 Model II: All Agents with Ambiguity

Now we suppose that all agents cannot learn the exact type, but only know whether $s \in I_H$ or $s \in I_L$. Thus all agents consider that the high type succeeds with probability $(1+s^*)/2$ and the low one does with probability $s^*/2$.

3.3.1 Separating Equilibria

In the separating equilibria, the angel and the VC offer share β and γ in (3.18) and (3.16), respectively, and thus the entrepreneur will have share θ_H in (3.20) if $s \in I_H$ and have θ_L in (3.20) if $s \in I_L$. Then the entrepreneur's expected utility is given by

$$U(t, \psi, \beta, \tilde{\gamma}(\beta)) = \begin{cases} \left(\dfrac{1+s^*}{2}\right)\theta_H A - c_H \psi & \text{if } t = I_H, \\ \\ \dfrac{s^*}{2}\theta_L A - c_L \psi & \text{if } t = I_L. \end{cases}$$

Similar to the benchmark model, the angel considers the following incentive compatibility constraints:

$$\frac{s^*}{2}\theta_L A \geq \frac{s^*}{2}\theta_H A - c_L \psi'_H \quad \text{and} \quad \left(\frac{1+s^*}{2}\right)\theta_H A - c_H \psi'_H \geq \left(\frac{1+s^*}{2}\right)\theta_L A.$$

Then we have the following proposition.

Proposition 3.5. *There are separating perfect Bayesian equilibria, where the patent levels of type I_L and type I_H entrepreneurs are given by*

$$\psi'_L = 0, \quad \text{and} \quad \psi'_H \in \left[\psi^{**}, \frac{K_1 + K_2}{s^* c_H}\right],$$

with

$$\psi^{**} \equiv \frac{K_1 + K_2}{(1+s^*)\, c_L},$$

the posterior belief and the offer of the angel are given by, respectively,

$$\mu(H|\psi) = \begin{cases} 0 & \text{if } \psi < \psi'_H, \\ 1 & \text{if } \psi \geq \psi'_H, \end{cases} \quad \text{and} \quad \tilde{\beta}(\psi) = \begin{cases} \beta_L & \text{if } \psi < \psi'_H, \\ \beta_H & \text{if } \psi \geq \psi'_H, \end{cases} \tag{3.29}$$

and the posterior belief and the offer of the VC are given by, respectively,

$$\pi(H|\beta) = \begin{cases} 0 & \text{if } \beta > \beta_H, \\ 1 & \text{if } \beta \leq \beta_H, \end{cases} \quad \text{and} \quad \tilde{\gamma}(\beta) = \begin{cases} \gamma_L & \text{if } \beta > \beta_H, \\ \gamma_H & \text{if } \beta \leq \beta_H. \end{cases} \tag{3.30}$$

3.3.2 Pooling Equilibria

In pooling equilibria, the angel and the VC offer share β and γ in (3.25) and (3.24), respectively, and thus the entrepreneur will have share θ_P in (3.26) regardless of his type. Then the entrepreneur's expected utility is given by

$$U\left(t, \psi, \beta, \tilde{\gamma}(\beta)\right) = \begin{cases} \left(\dfrac{1+s^*}{2}\right)\theta_P A - c_H \psi & \text{if } t = I_H, \\ \\ \dfrac{s^*}{2}\theta_P A - c_L \psi & \text{if } t = I_L. \end{cases}$$

Similar to the benchmark model, the angel considers the following incentive compatibility constraints:

$$\frac{s^*}{2}\theta_P A - c_L \psi'_P \geq \frac{s^*}{2}\theta_L A \quad \text{and} \quad \left(\frac{1+s^*}{2}\right)\theta_P A - c_H \psi'_P \geq \left(\frac{1+s^*}{2}\right)\theta_L A.$$

Then we have the following proposition.

Proposition 3.6. *There are pooling perfect Bayesian equilibria, where the patent level for the entrepreneur is given by*

$$\psi'_P \in \left[0, \frac{(K_1 + K_2)\, q}{(q + s^*)\, c_L}\right],$$

the posterior belief and the offer for the angel are given by, respectively,

$$\mu(H|\psi) = \begin{cases} 0 & \text{if } \psi \neq \psi_P \\ q & \text{if } \psi = \psi_P \end{cases} \quad \text{and} \quad \tilde{\beta}(\psi) = \begin{cases} \beta_L & \text{if } \psi \neq \psi_P, \\ \beta_P & \text{if } \psi = \psi_P, \end{cases} \tag{3.31}$$

and the posterior belief and the offer for the VC are given by, respectively,

$$\pi(H|\beta) = \begin{cases} 0 & \text{if } \beta \neq \beta_P \\ q & \text{if } \beta = \beta_P \end{cases} \quad \text{and} \quad \tilde{\gamma}(\beta) = \begin{cases} \gamma_L & \text{if } \beta \neq \beta_P, \\ \gamma_P & \text{if } \beta = \beta_P. \end{cases} \tag{3.32}$$

4 Refinements of Perfect Bayesian Equilibria

Now we refine our equilibria in Section 3 by imposing the Intuitive Criterion of Cho and Kreps (1987).[11] Let $T(\psi)$ be the set of types of the entrepreneur who might have chosen that patent level ψ, and for $T' \subset T(\psi)$, BR$(T', \psi, \tilde{\gamma})$ be the set of all pure-strategy best responses for the angel to patent level ψ and for beliefs $\mu(\cdot|\psi)$ such that $\mu\left(T'|\psi\right) = 1$:

$$\text{BR}\left(T', \psi, \tilde{\gamma}\right) = \bigcup_{\mu:\mu(T'|\psi)=1} \text{BR}\left(\mu, \psi, \tilde{\gamma}\right)$$

where

$$\text{BR}\left(\mu, \psi, \tilde{\gamma}\right) = \underset{\beta}{\text{argmax}} \sum_{t\in T'} \mu(t|\psi)\tilde{V}\left(t, \psi, \beta, \tilde{\gamma}(\beta)\right).$$

Let $U^*(t)$ be the entrepreneur's expected utility of type t in equilibrium. Note that $T\left(\psi\right) = T$ for any patent level $\psi \in [0, \infty)$. We define the Intuitive Criterion in our context of Definition 4.1.

DEFINITION 4.1. *A perfect Bayesian equilibrium fails the Intuitive Criterion if there exists $t' \in T(\psi) \setminus J\left(\psi, \tilde{\gamma}\right)$ with some ψ such that*

$$U^*\left(t'\right) < \min_{\beta\in BR(T(\psi)\setminus J(\psi,\tilde{\gamma}),\psi,\tilde{\gamma})} U\left(t', \psi, \beta, \tilde{\gamma}(\beta)\right), \tag{4.1}$$

where

$$J\left(\psi, \tilde{\gamma}\right) \equiv \left\{t \in T \,\middle|\, U^*(t) > \max_{\beta\in BR(T(\psi),\psi,\tilde{\gamma})} U\left(t, \psi, \beta, \tilde{\gamma}(\beta)\right)\right\}. \tag{4.2}$$

Pooling equilibria in each model are eliminated if we invoke the Intuitive Criterion. For each equilibrium patent level, the high type can find off-the-equilibrium message satisfying (4.1) which the low type would not send. Thus once if the angel correctly recognizes the type, the entrepreneur can increase his expected utility by deviating from the equilibria and thus all the pooling equilibria fails in the Intuitive Criterion.

Now consider the separating equilibria. In the benchmark model, all the high type's patent levels ψ_H^0 other than ψ^0 fail the Intuitive Criterion. Suppose that $\psi_H^0 \neq \psi^0$ and the angel observes an off-the-equilibrium patent level

$$\psi_H' = \frac{\psi^0 + \psi_H^0}{2}.$$

Then only the high type can be better off by sending off-the-equilibrium message ψ and thus we have $J\left(\psi, \tilde{\gamma}\right) = \{s_L\}$. Since the angel who observes off-the-equilibrium message ψ believe that the entrepreneur is high type with probability, (4.1) holds. Therefore, all ψ_H^0 where $\psi_H^0 \neq \psi^0$ do not survive the Incentive Criterion. Suppose that $\psi_H^0 = \psi^0$. It is clear that ψ^0 survives the Intuitive Criterion for off-the-equilibrium message $\psi \in \left(\psi^0, \infty\right)$. On the other hand, both types

[11] We provide a rigorous proof of the refinement in Appendix.

have incentive to send off-the-equilibrium message $\psi \in \left[0, \psi^0\right)$. Thus $J\left(\psi, \tilde{\gamma}\right) = \emptyset$. However, both types cannot have increased expected utility by sending off-the-equilibrium message ψ, if the angel perceives the entrepreneur as the low type. Thus (4.1) does not hold and the original equilibrium patent level ψ^0 survives the Intuitive Criterion.

Proposition 4.1. *In the benchmark model, there is a unique perfect Bayesian equilibrium that survives the Intuitive Criterion, the patent levels of type s_L and type s_H entrepreneurs are given by*

$$\psi_L^0 = 0 \quad \textit{and} \quad \psi_H^0 = \psi^0 = \frac{\left(s_H - s_L\right)\left(K_1 + K_2\right)}{s_H c_L},$$

the belief and offer of the angel are given by, respectively,

$$\mu(H|\psi) = \begin{cases} 0 & \textit{if } \psi < \psi^0, \\ 1 & \textit{if } \psi \geq \psi^0, \end{cases} \quad \textit{and} \quad \tilde{\beta}(\psi) = \begin{cases} \beta_L^0 & \textit{if } \psi < \psi^0, \\ \beta_H^0 & \textit{if } \psi \geq \psi^0, \end{cases}$$

and the belief and the offer of the VC are given by, respectively,

$$\pi(H|\beta) = \begin{cases} 0 & \textit{if } \beta > \beta_H^0, \\ 1 & \textit{if } \beta \leq \beta_H^0, \end{cases} \quad \textit{and} \quad \tilde{\gamma}(\beta) = \begin{cases} \gamma_L^0 & \textit{if } \beta > \beta_H^0, \\ \gamma_H^0 & \textit{if } \beta \leq \beta_H^0. \end{cases}$$

As mentioned in Subsection 3.2, the angel considers strong incentive compatibility constraints in order to ensure the separating the entrepreneur's types in Model I. Observing off-the-equilibrium message, the angel infers the types of the entrepreneur based on the incentive compatibility constraints. Applying similar argument in the benchmark model, we obtain the refined equilibrium of Model I given in Proposition 4.2

Proposition 4.2. *In Model I, there is a unique perfect Bayesian equilibrium that survives the Intuitive Criterion, the patent levels of type s_L and type s_H entrepreneurs are given by*

$$\psi_L = 0 \quad \textit{and} \quad \psi_H = \psi^* = \frac{2\left(K_1 + K_2\right)}{\left(1 + s^*\right) c_L},$$

where the posterior belief and the offer of the angel are given by, respectively,

$$\mu(H|\psi) = \begin{cases} 0 & \textit{if } \ \psi < \psi_H, \\ 1 & \textit{if } \ \psi \geq \psi_H, \end{cases} \quad \textit{and} \quad \tilde{\beta}(\psi) = \begin{cases} \beta_L & \textit{if } \ \psi < \psi_H, \\ \beta_H & \textit{if } \ \psi \geq \psi_H, \end{cases}$$

and the posterior belief and the offer of the VC are given by, respectively,

$$\pi(H|\beta) = \begin{cases} 0 & \textit{if } \ \beta > \beta_H, \\ 1 & \textit{if } \ \beta \leq \beta_H, \end{cases} \quad \textit{and} \quad \tilde{\gamma}(\beta) = \begin{cases} \gamma_L & \textit{if } \ \beta > \beta_H, \\ \gamma_H & \textit{if } \ \beta \leq \beta_H. \end{cases}$$

In model 2, the angel considers incentive compatibility constraints based on the expectation of s in each type since the entrepreneur also have ambiguous information about s. Similar arguments lead to a refined unique perfect Bayesian equilibrium of Model II in Proposition 4.3.

Proposition 4.3. *In Model II, there is a unique perfect Bayesian equilibrium that survives the Intuitive Criterion, the patent levels of type I_L and type I_H entrepreneurs are given by*

$$\psi_L' = 0 \quad \textit{and} \quad \psi_H^0 = \psi^* = \frac{K_1 + K_2}{(1 + s^*)\, c_L},$$

the belief and offer of the angel are given by, respectively,

$$\mu(H|\psi) = \begin{cases} 0 & \textit{if } \psi < \psi_H^0, \\ 1 & \textit{if } \psi \geq \psi_H^0, \end{cases} \quad \textit{and} \quad \tilde{\beta}(\psi) = \begin{cases} \beta_L & \textit{if } \psi < \psi_H^0, \\ \beta_H & \textit{if } \psi \geq \psi_H^0, \end{cases}$$

and the posterior belief and the offer of the VC are given by, respectively,

$$\pi(H|\beta) = \begin{cases} 0 & \textit{if } \beta > \beta_H, \\ 1 & \textit{if } \beta \leq \beta_H, \end{cases} \quad \textit{and} \quad \tilde{\gamma}(\beta) = \begin{cases} \gamma_L & \textit{if } \beta > \beta_H, \\ \gamma_H & \textit{if } \beta \leq \beta_H. \end{cases}$$

5 Comparative Statics

In this section, we characterize our PBE, which survives the Intuitive Criterion given in Proposition 4.1, Proposition 4.2, and Proposition 4.3. Here, we define the expected project return of the total investment by

$$\lambda^0(t) \equiv \begin{cases} \dfrac{s_H A}{K_1 + K_2}, & \text{if } t = s_H, \\ \\ \dfrac{s_L A}{K_1 + K_2}, & \text{if } t = s_L, \end{cases}$$

in the benchmark model and

$$\lambda(t) \equiv \begin{cases} \dfrac{(1 + s^*)\, A}{2\,(K_1 + K_2)}, & \text{if } t = I_H, \\ \\ \dfrac{s^* A}{2\,(K_1 + K_2)}, & \text{if } t = I_L, \end{cases}$$

in Model I and Model II.

Suppose that $(1 + s^*)/2 > s_H$. Then investors consider a higher success probability of the high type than his true success probability and thus we say that the success probability of the project is *overvalued* by investors. Similarly, if $s^*/2 > s_L$, then the project taken by the low type entrepreneur is overvalued by the investors. Regardless of $t \in I_H$ or $t \in I_L$, the project has a higher expected return of the total investment $K_1 + K_2$ under ambiguity (as in Model I and Model II) than without it (as in the benchmark model) if and only if the project is overvalued.

Proposition 5.1. *The following hold.*

(1) In each model, the entrepreneur's equity share increases in the expected project return of the total investment.

(2) The entrepreneur obtains a higher equity share in Model I and Model II than in the benchmark model if and only if the project is overvalued.

PROOF : (1) In the benchmark model, the entrepreneur's equity share is given by (3.5), which increases in $\lambda^0(t)$. In Model I and Model II, his equity share is given by (3.20), which increases in $\lambda(t)$. $\square$

(2) In Model I and Model II, since the equity share of the entrepreneur of the high type is given by (3.20), the share increases in s^*. Furthermore, we have $\theta_H > \theta_H^0$ if and only if $(1 + s^*)/2 > s_H$ and $\theta_L > \theta_L^0$ if and only if $s^*/2 > s_L$. $\blacksquare$

In each model, the equity share of the entrepreneur increases as the project has a higher expected return of the total investment. In particular, if investors face ambiguity, they expect that the high type has success probability $(1 + s^*)/2$ and the low type has success probability $s^*/2$ whether the entrepreneur faces ambiguity or not. Under Bertrand competition in investment markets, the angel and the VC ask lower equity shares as the success probabilities which they consider increase or required investments decrease, and it follows that the remaining share for the entrepreneur increases. From (3.5) and (3.20), we find that the entrepreneur's equity share when investors face ambiguity exceeds that in the benchmark model if and only if the project is overvalued.

Proposition 5.2. *In each model, the following hold.*

(1) The high type entrepreneur acquires a lower level of patent as the expected return of total investment increases with fixed A.

(2) The high type entrepreneur acquires a higher level of patent as the low type's marginal patents cost c_L decreases.

In each model, the high type entrepreneur spends a lower costs on filing patents as the project has a higher expected return of total investment. To be specific, as the required investments to launch and continue the project increase or the success probability which investors evaluate decreases, the high type entrepreneur should acquire more patents to signal his ability. To signal his ability, the high type entrepreneur should show a higher degree of commitment by spending his own money as the expected project return of total investment decreases. From (2) of Proposition 5.2, we observe that the optimal equilibrium patent level of the high type in each model depends on the marginal patents cost c_L of the low type. As the marginal cost c_L of the low type entrepreneur increases, the high type can separate himself and signal his type with a lower patent cost. Consequently, the high type can save his patent cost as the low type acquires patents more inefficiently.

Proposition 5.3. *The following hold.*

(1) The high type entrepreneur acquires a higher level of patent in Model I than in the benchmark model and Model II.

(2) The high type entrepreneur acquires a higher level of patent in Model II than in the benchmark model if and only if

$$s^* < \frac{s_L}{s_H - s_L}. \tag{5.1}$$

The high type entrepreneur should spend more money to signal his ability in Model I than in the benchmark model and in Model II, i.e., $\psi^* > \psi^0$ and $\psi^* > \psi^{**}$. In Model I, although he resolves ambiguity, the high type entrepreneur cannot reduce patents cost and even pays more for filing patents than in other models since the angel under ambiguity considers the extremely conservative incentive compatibility constraints, which hold for all possible s_L and s_H.

In the benchmark model and Model II, on the other hand, the entrepreneur and investors have same information about the type of the entrepreneur in the refined equilibria. Then the angel considers incentive compatibility constraints based on her belief about success probability. The optimal patent level may be higher or lower in Model II than in the benchmark model depending on investors' evaluation about the success probability. In Model II, however, if investors set a sufficiently low value on the project such that (5.1) holds, the high type entrepreneur should acquire more patents than in the benchmark.

Proposition 5.4. *The following hold.*

(1) The expected utilities of the low type entrepreneur are higher in Model I and Model II than in the benchmark model if and only if $s^*/2 > s_L$.

(2) The expected utility of the high type entrepreneur is higher in Model I than in the benchmark model if and only if

$$s^* > \frac{c_H\left(s_H + s_L\right) - \left(1 - 2s_H\right) c_L s_H}{\left(s_H - s_L\right) c_H + c_L s_H}. \tag{5.2}$$

(3) The expected utility of the high type entrepreneur is higher in Model II than in the benchmark model if

$$s^* > \max\left\{2s_H - 1, \frac{s_L}{s_H - s_L}\right\} \tag{5.3}$$

holds.

The low type entrepreneur does not spend money on filing patents in all the refined PBE in Section 4. Thus his expected utility only depends on the expected income. From (3.5) and (3.20), he takes a higher equity share when investors face ambiguity than when no one has ambiguous information if and only if the project taken by the low type is overvalued. Furthermore, compared to the benchmark model, if his success probability is overvalued, the low type entrepreneur considers a higher success probability of the project in Model II. Under the overvaluation, consequently, the low type has a higher expected utility when ambiguity is present in the investment markets than when there is no ambiguity.

Now we compare the expected utilities of the high type entrepreneur in Model I and Model II with the expected utility in the benchmark model. Note that (2) and (3) of Proposition 5.4 show that the high type entrepreneur has a higher expected utility in Model I and Model II than that in the benchmark model if investors set a sufficiently high evaluation on the success probability of the project. The expected utility of the high type depends on both his expected income and patents costs. In Model I, the high type spends more money on acquiring patents than in the benchmark model from (1) of Proposition 5.3. On the other hand, his expected income increases in threshold probability s^* since his equity share θ_H increases in s^* from (3.20). If threshold probability s^* is sufficiently high such that (5.2) holds, the increase in the expected income dominates that in the patents cost. This follows from (2) of Proposition 5.4. Now suppose that if s^* is high enough such that (5.3) holds. In Model II, the high type pays less patents cost than that in the benchmark model from (2) of Proposition 5.3. Furthermore, his expected income is also higher in Model II than that in the benchmark since

$$\left(\frac{1+s^*}{2}\right)\theta_H A = \frac{(1+s^*)\,A}{2} - (K_1+K_2) > s_H A - (K_1+K_2) = \theta_H^0 s_H A.$$

This follows from (3) of Proposition 5.4.

Proposition 5.5. *The following hold.*

(1) The expected utility of the low type entrepreneur is higher in Model I than in Model II if and only if $s_L > s^*/2$.

(2) The expected utility of the high type entrepreneur is higher in Model I than in Model II if and only if

$$s_H > \frac{1+s^*}{2} + \frac{c_H\,(K_1+K_2)}{c_L\,[(1+s^*)\,A - 2\,(K_1+K_2)]} \tag{5.4}$$

holds.

Each claim in Proposition 5.5 shows that the entrepreneur can be better off by resolving ambiguity when the project is (sufficiently) undervalued by ambiguous agents. Recall that the entrepreneur's equity shares in Model I and Model II are equivalent. Since the low type does not expense patents cost, the difference of his expected utilities between Model I and Model II only depends on success probability which he considers. Therefore his expected utility is higher when only investors face ambiguity than when all agents face it if and only if his success probability is undervalued.

Now we consider the high types in Model I and Model II. From (1) of Proposition 5.3, we know that the optimal patent level of the high type is higher in Model I than that in Model II.

On the other hand, the expected income in Model I may be higher or lower than that in Model II depending on s_H and s^*. However, if the success probability s_H of the high type is sufficiently high such that (5.4) holds, an increase in the expected income dominates that in patents cost and therefore the high type has a higher expected utility in Model I than in Model II. Note that the right-hand side in (5.4) consists of market evaluation $(1 + s^*)/2$ of the high type's success probability and an additional value. Contrary to the low type, the high type can increase his expected utility by resolving ambiguity only when his success probability sufficiently exceeds investors' evaluation because he should pay more patents cost to signal his type in Model I than in Model II.

6 Concluding Remarks

Agents in early-stage investments usually face ambiguity due to the lack of track records. To examine the effects of ambiguity, we provide three different two-stage financing models in view of the degrees of ambiguity: (1) no ambiguity, (2) only investors face ambiguity, (3) all agents face ambiguity. In each model, we derive the perfect Bayesian equilibria of the signaling game and refine them into a unique equilibrium in view of the Intuitive Criterion of Cho and Kreps (1987) and characterize it. We find that the entrepreneur with high ability should spend more money in the model where only investors face ambiguity than other models. It is because, to avoid failing to separate the entrepreneur's type, somewhat strongly incentive compatibility constraints are considered by the angel who faces ambiguity with knowing that the entrepreneur resolves it. This implies that the entrepreneur cannot save patents costs by resolving ambiguity. We also find that the expected utility of the entrepreneur increases when investors face ambiguity than otherwise if the project is sufficiently overvalued in investment markets.

Future research can proceed in two possible directions. First, we may consider the case where the patent is productive and adds value to the project as in Conti, Thursby, and Rothaermel (2013) and Conti, Thursby, and Thursby (2013). Second, we can construct a model of ambiguity-averse preferences. For instance, we can consider a model of early-state financing markets based on a concave distortion function rather than a linear one in Klibanoff, Marinacci, and Mukerji (2005) or consider a model in which investors have extremely conservative beliefs about the probability distribution of his returns and have maxmin expected utility of Gilboa and Schmeidler (1989).

Appendix

PROOF OF PROPOSITION 3.1: If the belief of the VC is given by π in (3.8), the VC's optimal offer is $\tilde{\beta}$ in (3.8). Similarly, we obtain the belief and offer of the angel in (3.7). Under the belief μ in

(3.7), the low type chooses patent level of zero and the high type chooses patent level ψ_H^0. The incentive compatibility constraints for the low type entrepreneur and high type entrepreneur are given by

$$\theta_L^0 s_L A \geq \theta_H^0 s_L A - c_L \psi_H^0 \quad \text{and} \quad \theta_H^0 s_H A - c_H \psi_H^0 \geq \theta_L^0 s_H A,$$

respectively. Then we have

$$\psi_H^0 \in \left[\frac{\left(\theta_H^0 - \theta_L^0\right) s_L A}{c_L}, \frac{\left(\theta_H^0 - \theta_L^0\right) s_H A}{c_H} \right]. \tag{A.1}$$

By (3.5), A.1 can be rewritten as

$$\psi_H^0 \in \left[\frac{(s_H - s_L)(K_1 + K_2)}{s_H c_L}, \frac{(s_H - s_L)(K_1 + K_2)}{s_L c_H} \right]. \qquad \blacksquare$$

PROOF OF PROPOSITION 3.2: If the belief of the VC is given by π in (3.14), the VC's optimal offer is $\tilde{\gamma}$ in (3.14). Similarly, we obtain the belief and offer of the angel in (3.13). To hold patent level ψ_P^0, incentive compatibility constraint for the type I_L entrepreneur is

$$\theta_P^0 s_L A - c_L \psi_P^0 \geq \theta_L^0 s_L A$$

and that for the type I_H entrepreneur is

$$\theta_P^0 s_H A - c_H \psi_P^0 \geq \theta_L^0 s_H A$$

Therefore,

$$\psi_P^0 \in \left[0, \frac{\left(\theta_P^0 - \theta_L^0\right) s_L A}{c_L} \right]. \tag{A.2}$$

Substituting θ_P and θ_L into (A.4), we have

$$\psi_P^0 \in \left[0, \frac{q\,(s_H - s_L)(K_1 + K_2)}{[(1-q)s_L + q s_H]\, c_L} \right]. \qquad \blacksquare$$

PROOF OF PROPOSITION 3.5: If the belief of the VC is given by π in (3.30), the VC's optimal offer is $\tilde{\gamma}$ in (3.30). Similarly, we obtain the belief and offer of the angel in (3.29). Under the belief μ in (3.29) of the angel, the low type chooses patent level of zero and the high type chooses patent level ψ_H^0. The incentive compatibility constraints for the low type entrepreneur and high type entrepreneur are given by

$$\frac{s^*}{2}\theta_L A \geq \frac{s^*}{2}\theta_H A - c_L \psi_H \quad \text{and} \quad \left(\frac{1+s^*}{2}\right)\theta_H A - c_H \psi_H \geq \left(\frac{1+s^*}{2}\right)\theta_L A,$$

respectively. Then we have

$$\psi_H^0 \in \left[\frac{(\theta_H - \theta_L)\, s^* A}{2c_L}, \frac{(1+s^*)(\theta_H - \theta_L)\, A}{2c_H} \right]. \tag{A.3}$$

Substituting θ_L and θ_H into (A.3), we obtain

$$\psi_H^0 \in \left[\frac{K_1 + K_2}{(1+s^*)\, c_L}, \frac{K_1 + K_2}{s^* c_H}\right]. \qquad \blacksquare$$

Proof of Proposition 3.6: If the belief of the VC is given by π in (3.32), the VC's optimal offer is $\tilde{\gamma}$ in (3.32). Similarly, we obtain the belief and offer of the angel in (3.31). To hold patent level ψ_P', incentive compatibility constraint for the type I_L entrepreneur is

$$\frac{s^*}{2}\theta_P A - c_L \psi_P \geq \frac{s^*}{2}\theta_L A$$

and that for the type I_H entrepreneur is

$$\left(\frac{1+s^*}{2}\right)\theta_P A - c_H \psi_P \geq \left(\frac{1+s^*}{2}\right)\theta_H A$$

Therefore,

$$\psi_P' \in \left[0, \frac{(\theta_P - \theta_L)\, s^* A}{2c_L}\right]. \tag{A.4}$$

Substituting θ_P and θ_L into (A.4), we have

$$\psi_P' \in \left[0, \frac{(K_1 + K_2)\, q}{(q+s^*)\, c_L}\right]. \qquad \blacksquare$$

Proof of Proposition 4.1 Suppose that only the entrepreneur can learn his exact type s while investors have the exact information about possible types.

(1) Separating Equilibria

Consider the separating equilibria in Proposition 3.1.

(Case 1) Separating equilibria where $\psi_H \neq \psi^0$

Consider an off-the-equilibrium patent level

$$\psi = \frac{\psi^0 + \psi_H^0}{2}.$$

Since

$$U^*(s_L) = \theta_L^0 s_L A > \theta_H^0 s_L A - c_L \psi = \max_{\beta \in \mathrm{BR}(T(\psi),\psi,\tilde{\gamma})} U(s_L, \psi, \beta, \gamma),$$

the low type has no incentive to deviate from the equilibrium patent level ψ_L^0.

On the other hand, the high type can increase his expected utility by sending ψ if the angel perceives him as the high type since

$$U^*(s_H) = \theta_H^0 s_H A - c_H \psi_H < \theta_H^0 s_H A - c_H \psi = \max_{\beta \in \mathrm{BR}(T(\psi),\psi,\tilde{\gamma})} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)).$$

Thus $T(\psi) \setminus J(\psi, \tilde{\gamma}) = \{H\}$. Observe ψ, the angel believes the entrepreneur is the high type with probability 1, i.e., $\mu(H|\psi) = 1$.

Now we check whether inequality (4.1) holds or not. For the off-the-equilibrium patent level ψ, since we have

$$U^*(s_H) = \theta_H^0 s_H A - c_H \psi_H < \theta_H^0 s_H A - c_H \psi = \min_{\beta \in \text{BR}(\{H\}, \psi)} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)).$$

Thus (4.1) holds and the original equilibrium patent level ψ_H fails the Intuitive Criterion.

(Case 2) Separating equilibria where $\psi_H^0 = \psi^0$

First, consider any off-the-equilibrium patent level $\psi \in (\psi^0, \infty)$. Since

$$\begin{aligned} U^*(s_H) &= \theta_H^0 s_H A - c_H \psi^0 > \theta_H^0 s_H A - c_H \psi = \max_{\beta \in \text{BR}(T(\psi), \psi, \beta, \tilde{\gamma})} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)), \\ U^*(s_L) &= \theta_H^0 s_L A > \theta_H^0 s_L A - c_L \psi = \max_{\beta \in \text{BR}(T(\psi), \psi, \tilde{\gamma})} U(s_L, \psi, \beta, \tilde{\gamma}(\beta)), \end{aligned}$$

both types has no incentive to deviate from equilibrium patent levels. Thus $T(\psi) \setminus J(\psi, \tilde{\gamma}) = \emptyset$. Therefore the original equilibrium patent level ψ^0 survives the Intuitive Criterion.

Second, consider an off-the-equilibrium patent level $\psi \in [0, \psi^0)$. The high type can increase his utility if the angel correctly identifies his type since

$$U^*(s_H) = \theta_H^0 s_H A - c_H \psi^0 < \theta_H^0 s_H A - c_H \psi = \max_{\beta \in \text{BR}(T(\psi), \psi, \tilde{\gamma})} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)).$$

The low type also can be better off if the angel misunderstands his type since

$$U^*(s_L) = \theta_L^0 s_L A < \theta_H^0 s_L A - c_L \psi = \max_{\beta \in \text{BR}(T(\psi), \psi, \tilde{\gamma})} U(s_L, \psi, \beta, \tilde{\gamma}(\beta)).$$

Thus $T(\psi) \setminus J(\psi) = T$. Now we check whether inequality (4.1) holds or not. Each type of entrepreneur obtains the minimum expected utility when the angel believes that he is the low type. Note that

$$\begin{aligned} U^*(s_L) &= \theta_L^0 s_L A > \theta_L^0 s_L A - c_L \psi = \min_{\beta \in \text{BR}(T, \psi)} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)), \\ U^*(s_H) &= \theta_H^0 s_H A - c_H \psi^0 > \theta_L^0 s_H A - c_H \psi = \min_{\beta \in \text{BR}(T, \psi)} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)). \end{aligned}$$

Thus, for any off-the-equilibrium message ψ, (4.1) holds and the original equilibrium survives the Intuitive Criterion.

(2) Pooling Equilibrium

Consider the pooling equilibria in Proposition 3.2. Solving

$$\theta_H^0 s_L A - c_L \psi_P^1 = \theta_P^0 s_L A - c_L \psi_P \quad \text{and} \quad \theta_H^0 s_H A - c_H \psi_P^2 = \theta_P^0 s_H A - c_H \psi_P^0$$

for ψ_P^1 and ψ_P^2, we have

$$\psi_P^1 = \frac{s_L(1-q)\,(s_H - s_L)\,(K_1+K_2)}{c_L s_H\,[(1-q)s_L + q s_H]} + \psi_P^0 \quad \text{and} \quad \psi_P^2 = \frac{(1-q)\,(s_H - s_L)\,(K_1+K_2)}{c_L\,[(1-q)s_L + q s_H]} + \psi_P^0.$$

Consider an off-the-equilibrium message

$$\psi = \frac{\psi_P^1 + \psi_P^2}{2}.$$

Since

$$U^*\left(S_L\right) = \theta_P^0 s_L A - c_L \psi_P^0 > \theta_H s_L A - c_L \psi = \max_{\beta \in \mathrm{BR}(T(\psi),\psi,\tilde{\gamma})} U\left(s_L, \psi, \beta, \tilde{\gamma}(\beta)\right),$$

the low type cannot increase his utility by deviate to ψ from ψ_P^0. The high type can increase his expected utility if the angel perceives him as the high type since

$$U^*\left(s_H\right) = \theta_P^0 s_H A - c_H \psi_P^0 < \theta_H^0 s_H A - c_H \psi = \max_{\beta \in \mathrm{BR}(T(\psi),\psi,\tilde{\gamma})} U\left(s_H, \psi, \beta, \tilde{\gamma}(\beta)\right).$$

Thus $T(\psi) \setminus J(\psi) = \{s_H\}$. Angels who observe ψ believe that the entrepreneur is the high type and thus we have

$$U^*\left(s_H\right) = \theta_P^0 s_H A - c_H \psi_P^0 < \theta_H^0 s_H A - c_H \psi = \min_{\beta \in \mathrm{BR}(\{H\},\psi,\tilde{\gamma})} U\left(s_H, \psi, \beta, \tilde{\gamma}(\beta)\right).$$

Therefore, the pooling equilibria in Proposition 3.2 fail the Intuitive Criterion. ■

Proof of Proposition 4.2 Suppose that only the entrepreneur can learn his exact type s while investors have ambiguous information about possible types.

(1) Separating Equilibria

Consider the separating equilibria in Proposition 3.3.

(Case 1) Separating equilibria where $\psi_H \neq \psi^*$

Consider an off-the-equilibrium patent level

$$\psi = \frac{\psi^* + \psi_H}{2}.$$

Since

$$U^*\left(s_L\right) = \theta_L s_L A > \theta_H s_L A - c_L \psi = \max_{\beta \in \mathrm{BR}(T(\psi),\psi,\tilde{\gamma})} U\left(s_L, \psi, \beta, \gamma\right),$$

the low type has no incentive to deviate from the equilibrium patent level ψ_L.

On the other hand, the high type can increase his expected utility by sending ψ if the angel perceives him as the high type since

$$U^*\left(s_H\right) = \theta_H s_H A - c_H \psi_H < \theta_H s_H A - c_H \psi = \max_{\beta \in \mathrm{BR}(T(\psi),\psi,\tilde{\gamma})} U\left(s_H, \psi, \beta, \tilde{\gamma}(\beta)\right).$$

Thus $T(\psi) \setminus J(\psi, \tilde{\gamma}) = \{H\}$. Observe ψ, the angel believes the entrepreneur is the high type with probability 1, i.e., $\mu(H|\psi) = 1$.

Now we check whether inequality (4.1) holds or not. For the off-the-equilibrium patent level ψ, since we have

$$U^*(s_H) = \theta_H s_H A - c_H \psi_H < \theta_H s_H A - c_H \psi = \min_{\beta \in \text{BR}(\{H\}, \psi)} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)).$$

Thus (4.1) holds and the original equilibrium patent level ψ_H fails the Intuitive Criterion.

(Case 2) Separating equilibria where $\psi_H^0 = \psi^*$

First, consider any off-the-equilibrium patent level $\psi \in (\psi^*, \infty)$. Since

$$\begin{aligned} U^*(s_H) &= \theta_H s_H A - c_H \psi^* > \theta_H s_H A - c_H \psi = \max_{\beta \in \text{BR}(T(\psi), \psi, \beta, \tilde{\gamma})} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)), \\ U^*(s_L) &= \theta_H s_L A > \theta_H s_L A - c_L \psi = \max_{\beta \in \text{BR}(T(\psi), \psi, \tilde{\gamma})} U(s_L, \psi, \beta, \tilde{\gamma}(\beta)), \end{aligned}$$

both types has no incentive to deviate from equilibrium patent levels. Thus $T(\psi) \setminus J(\psi, \tilde{\gamma}) = \emptyset$. Therefore the original equilibrium patent level ψ^* survives the Intuitive Criterion.

Second, consider an off-the-equilibrium patent level $\psi \in [0, \psi^*)$. The high type can increase his utility if the angel correctly identifies his type since

$$U^*(s_H) = \theta_H s_H A - c_H \psi^* < \theta_H s_H A - c_H \psi = \max_{\beta \in \text{BR}(T(\psi), \psi, \tilde{\gamma})} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)).$$

The low type also can be better off if the angel misunderstands his type since

$$U^*(s_L) = \theta_L s_L A < \theta_H s_L A - c_L \psi = \max_{\beta \in \text{BR}(T(\psi), \psi, \tilde{\gamma})} U(s_L, \psi, \beta, \tilde{\gamma}(\beta))$$

for ψ such that

$$\psi < \frac{2(K_1 + K_2) s_L}{(1 + s^*) s^* c_L} < \psi^*.$$

Thus $T(\psi) \setminus J(\psi) = T$. Now we check whether inequality (4.1) holds or not. Each type of entrepreneur obtains the minimum expected utility when the angel believes that he is the low type. Note that

$$\begin{aligned} U^*(s_L) &= \theta_L s_L A > \theta_L s_L A - c_L \psi = \min_{\beta \in \text{BR}(T, \psi)} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)), \\ U^*(s_H) &= \theta_H s_H A - c_H \psi^* > \theta_L s_H A - c_H \psi = \min_{\beta \in \text{BR}(T, \psi)} U(s_H, \psi, \beta, \tilde{\gamma}(\beta)). \end{aligned}$$

Thus, for any off-the-equilibrium message ψ, (4.1) holds and the original equilibrium survives the Intuitive Criterion.

(2) Pooling Equilibrium

Consider the pooling equilibrium in Proposition 3.4. Solving

$$\theta_H s_L A - c_L \psi_P^1 = \theta_P s_L A - c_L \psi_P \quad \text{and} \quad \theta_H s_H A - c_H \psi_P^2 = \theta_P s_H A - c_H \psi_P$$

for ψ_P^1 and ψ_P^2, we have

$$\psi_P^1 = \frac{2(1-q)\left(K_1+K_2\right)s_L}{\left(1+s^*\right)\left(q+s^*\right)c_L} \quad \text{and} \quad \psi_P^2 = \frac{2(1-q)\left(K_1+K_2\right)s_H}{\left(1+s^*\right)\left(q+s^*\right)c_L}.$$

Consider an off-the-equilibrium message

$$\psi = \frac{\psi_P^1 + \psi_P^2}{2}.$$

Since

$$U^*\left(S_L\right) = \theta_P s_L A - c_L \psi_P > \theta_H s_L A - c_L \psi = \max_{\beta \in \mathrm{BR}(T(\psi),\psi,\tilde{\gamma})} U\left(s_L, \psi, \beta, \tilde{\gamma}(\beta)\right),$$

the low type cannot increase his utility by deviate to ψ from ψ_P. The high type can increase his expected utility if the angel perceives him as the high type since

$$U^*\left(s_H\right) = \theta_P s_H A - c_H \psi_P < \theta_H s_H A - c_H \psi = \max_{\beta \in \mathrm{BR}(T(\psi),\psi,\tilde{\gamma})} U\left(s_H, \psi, \beta, \tilde{\gamma}(\beta)\right).$$

Thus $T(\psi) \setminus J(\psi) = \{s_H\}$. Angels who observe ψ believe that the entrepreneur is the high type and thus we have

$$U^*\left(s_H\right) = \theta_P s_H A - c_H \psi_P < \theta_H s_H A - c_H \psi = \min_{\beta \in \mathrm{BR}(\{H\},\psi,\tilde{\gamma})} U\left(s_H, \psi, \beta, \tilde{\gamma}(\beta)\right).$$

Therefore, the pooling equilibrium in Proposition 3.4 fails the Intuitive Criterion. ■

PROOF OF PROPOSITION 5.3 (1) We have

$$\psi^* - \psi^0 = \frac{\left[\left(1-s^*\right)s_H + \left(1+s^*\right)s_L\right]\left(K_1+K_2\right)}{\left(1+s^*\right)c_L s_H} > 0.$$

Furthermore, $\psi^* > \psi^{**}$ by Propositions 4.2-4.3, the claim holds. □

(2) By Proposition 4.1 and Proposition 4.3, we have

$$\psi^{**} - \psi^0 = \frac{\left[\left(1+s^*\right)s_L - s^* s_H\right]\left(K_1+K_2\right)}{\left(1+s^*\right)c_L s_H},$$

which is greater than zero if and only if (5.1) holds. This implies the claim. ■

PROOF OF PROPOSITION 5.4 (1) Note that

$$\begin{aligned} U^*\left(s_L, 0, \beta_L, \gamma_L\right) - U^*\left(s_L, 0, \beta_L^0, \gamma_L^0\right) &= \frac{\left(s^* - 2s_L\right)\left(K_1+K_2\right)}{s^*} > 0 \\ U^*\left(L, 0, \beta_L, \gamma_L\right) - U^*\left(s_L, 0, \beta_L^0, \gamma_L^0\right) &= \frac{1}{2}\left(s^* - 2s_L\right)A > 0 \end{aligned}$$

if and only if $s^*/2 > s_L$. Thus the low type entrepreneur obtains higher expected utility in Model I and Model II than in the benchmark model if and only if $s^*/2 > s_L$. $\square$

(2) We have

$$\begin{aligned}&U^*\left(s_H,0,\beta_H,\gamma_H\right)-U^*\left(s_H,0,\beta_H^0,\gamma_H^0\right)\\=\;&\frac{\left[\left(1+s^*-2s_H\right)c_Ls_H-\left\{\left(1-s^*\right)s_H+\left(1+s^*\right)s_L\right\}c_H\right]\left(K_1+K_2\right)}{\left(1+s^*\right)c_Ls_H},\end{aligned}$$

which increases in s^*. Since $U^*\left(s_H,0,\beta_H,\gamma_H\right)-U^*\left(s_H,0,\beta_H^0,\gamma_H^0\right)$ becomes zero if and only if

$$s^*=\frac{c_H\left(s_H+s_L\right)-\left(1-2s_H\right)c_Ls_H}{\left(s_H-s_L\right)c_H+c_Ls_H},$$

the high type entrepreneur has a higher expected utility in Model I than in the benchmark if and only if (5.2) holds. $\square$

(3) We have

$$\begin{aligned}&U^*\left(H,0,\beta_H,\gamma_H\right)-U^*\left(s_H,0,\beta_H^0,\gamma_H^0\right)\\=\;&\frac{\left(1+s^*\right)\left(1+s^*-2s_H\right)c_Ls_HA-2\left[\left(1+s^*\right)s_L-s^*s_H\right]\left(K_1+K_2\right)c_H}{2\left(1+s^*\right)c_Ls_H},\end{aligned}$$

which is greater than zero if $1+s^*-2s_H>0$ and $\left(1+s^*\right)s_L-s^*s_H<0$. Therefore, if (5.3) holds, we have $U^*\left(H,0,\beta_H,\gamma_H\right)>U^*\left(s_H,0,\beta_H^0,\gamma_H^0\right)$. $\blacksquare$

PROOF OF PROPOSITION 5.5 (1) The expected utilities of the low type entrepreneur in Model I and Model II are given by

$$\begin{aligned}U^*\left(s_L,0,\beta_L,\gamma_L\right)&=\theta_Ls_LA=\frac{\left[s^*A-2\left(K_1+K_2\right)\right]s_L}{s^*},\\U^*\left(L,0,\beta_L,\gamma_L\right)&=\frac{s^*}{2}\theta_LA=\frac{s^*A}{2}-\left(K_1+K_2\right),\end{aligned}$$

respectively. It follows that $U^*\left(s_L,0,\beta_L,\gamma_L\right)>U^*\left(L,0,\beta_L,\gamma_L\right)$ if and only if $s_L>s^*/2$. $\square$

(2) The expected utilities of the high type entrepreneur in Model I and Model II are given by

$$\begin{aligned}U^*\left(s_H,0,\beta_H,\gamma_H\right)&=\theta_Hs_HA-c_H\psi^*=s_HA-\frac{2\left(c_H+c_Ls_H\right)\left(K_1+K_2\right)}{\left(1+s^*\right)c_L},\\U^*\left(H,0,\beta_H,\gamma_H\right)&=\left(\frac{1+s^*}{2}\right)\theta_HA-c_H\psi^{**}=\frac{\left(1+s^*\right)A}{2}-\frac{\left[\left(1+s^*\right)c_L+c_H\right]\left(K_1+K_2\right)}{\left(1+s^*\right)c_L},\end{aligned}$$

respectively. We have

$$U^*\left(s_H,0,\beta_H,\gamma_H\right)-U^*\left(H,0,\beta_H,\gamma_H\right)=-\frac{\left(1+s^*-2s_H\right)\left[\left(1+s^*\right)A-2\left(K_1+K_2\right)\right]c_L+2\left(K_1+K_2\right)c_H}{2\left(1+s^*\right)c_L},$$

which is higher than zero if and only if

$$s_H>\frac{1+s^*}{2}+\frac{c_H\left(K_1+K_2\right)}{\left[\left(1+s^*\right)A-2\left(K_1+K_2\right)\right]c_L}. \blacksquare$$

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