Analytical approximations of American call options with a discrete dividend

Abstract

While piecewise geometric Brownian motion is a stochastic process that can effectively incorporate discrete dividends into stock prices without losing consistency, the process results in the lack of closed-form solutions for option prices. We aim to resolve this by providing analytical approximation formulas for American call option prices under this process. Our approximations include lower and upper bounds of the option prices. The additional numerical analysis indicates that the lower bound method is very efficient and it is possible to reduce the pricing error by considering additional terms.

Keywords: American call option; Discrete dividend; Approximation; Lower bound; Upper bound

1. Introduction

After Black and Scholes (1973) proposed option pricing theory, many studies attempted to introduce more realistic assumptions about dividends. One solution is a continuous constant yield of dividends. Under this assumption, Merton (1973) provides a pricing formula for European options and Kim (1990), Jacka (1991), Carr, Jarrow, and Myneni (1992), and Jamshidian (1992) provide a pricing formula for American options in an implicit form. However, Alan, Graham, Campbell, and Harvey (2004) noted that firms provide dividends in discrete rather than continuous flows and CEOs are reluctant to change the size of dividends,[1] and thus discrete constant dividends are an alternative.

To incorporate discrete constant dividends into European option pricing, Black (1975) suggests using the stock price minus the present value of dividends instead of the stock price itself in Black and Scholes' (1973) proposed formula. When firms escrow of the dividend's value and run the business with the rest, Black's (1975) method provides the exact value of European options. Furthermore, under this escrow model, we have the analytical pricing formula for American call options from Roll (1977), Geske (1979a), and Whaley (1981). However, the escrow model is flawed because the stock price dynamics for options with a maturity before a dividend date is inconsistent with the stock price dynamics for options with a maturity after the dividend date.

Therefore, we assume that stock price follows piecewise geometric Brownian motion, a variant of geometric Brownian motion except the ex-dividend date, following others, including Areal and Rodrigues (2014). Although this model is free of inconsistency, it still lacks an

[1] Areal and Rodrigues (2014) show that even index options can have a discreteness pattern using FTSE-100 options.

analytical closed-form solution. To overcome this problem, a lot of studies develop analytical approximation formulas for European option prices (M. Bos & Vandermark, 2002; R. Bos, Gairat, & Shepeleva, 2003; Dai & Lyuu, 2009; de Matos, Dilão, & Ferreira, 2009; Etoré & Gobet, 2012; Sahel & Gocsei, 2011; Veiga & Wystup, 2009). However, studies consider binomial trees[2] for American option pricing and analytical approximation formulas are rare.[3] This is likely because early exercise makes hard to determine the present value of the expected payoff. However, as Merton (1973) shows, the optimal exercise timing is limited to just before the ex-dividend date for call options.[4] From this point of view, Haug, Haug, and Lewis (2003) provide an exact pricing formula for American call options. Nevertheless, the formula has a practical problem in that using it requires numerical integration because their solution is represented in an integral form.

In this study, we provide analytical approximation formulas for American call option based on the integral form under piecewise geometric Brownian motion. Our work differs from other studies using the same assumption at least in two respects. First, and most importantly, we investigate the analytical approximations of American call options and examine European call option as a special case, while almost all analytical approximations in the literature cover only

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[2] For example, Vellekoop and Nieuwenhuis (2006) and Areal and Rodrigues (2013) provide an efficient algorithm to determine the value of American options under these assumptions.

[3] While barrier options are not American options, Dai and Chiu (2014) develop approximating analytical formulae. For continuous dividends, Barone-Adesi and Whaley (1987) and G. Chang, Kang, Kim, and Kim (2007) develop analytical approximation formulas for American options and American exotic options by approximating partial differential equation.

[4] Therefore, an American call option on a non-dividend paying stock is equivalent to a European call option. Due to the symmetric relationship between American call and put options, Miao, Lee, and Chao (2014) show that an American put option is equivalent to a European put option when the interest rate is zero.

European options, which is a very popular area of study most likely because of the efficiency and accuracy of the approximation method. [5] Therefore, attempting an analytical approximation itself is a contribution in terms of pricing American call options. Second, our solution is equal to the exact price when the size of the dividend is proportional to the stock price, while results from the binomial tree never match the exact price of options for any circumstance. The numerical analysis demonstrates the efficiency of our method.

The remainder of the paper is organized as follows. Section 2 presents the models for the stock price and dividend, and we recall the option prices in the integral form. Section 3 finds the analytical approximations that include upper and lower bounds. Section 4 compares the accuracy and speed of the methods via numerical analysis. Section 5 concludes this study.

2. Model

Assume a risk-free asset with a rate of return *r*. We consider a stock with price S_t at time *t* and an American call option on the stock with exercise price *K* and maturity *T*. The stock will distribute dividend D_τ at τ. The stock price then has the following dynamics under the risk-neutral probability measure:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q, \ \ f \ \ t \neq \tau \tag{1}$$

$$S_\tau = S_{\tau^-} - D_\tau \tag{2}$$

[5] Analytical approximation formulas are also used for exotic option pricing. For example, moment matching, conditional expectation, and Taylor expansions exist for basket option pricing (J.-J. Chang, Chen, & Wu, 2012; Curran, 1994; N. Ju, 2002; Posner & Milevsky, 1998).

The dividend amount is generally linked to the stock price, although it is not that sensitive. Therefore, we assume an affine dividend in terms of the stock price:[6]

$$D_\tau = \min\{yS_{\tau^-} + \delta, S_{\tau^-}\} \tag{3}$$

for $y \geq 0$ and $\delta \geq 0$.

Unlike European options, we rarely find analytical approximation for American options, probably due to the early exercise. However, in the case of the call options, the optimal early exercise is limited to the ex-dividend date. In this spirit, Haug et al. (2003) provide pricing for American call options in an integral form. To summarize their proposal, the stock price just before the ex-dividend date (S_{τ^-}) follows the common log-normal distribution. In addition, the option value is given in a closed form at τ^-.[7] We can therefore obtain the option price by discounting the expected value at τ^-, as follows:

Remark. Integral form of American call option price (Haug *et al*. 2003)

$$\begin{aligned} C_A(S_0, 0; D_\tau, \tau) &= e^{-r\tau} \int_{\frac{\delta}{1-y}}^{S^*} C_E(S(1-y) - \delta, \tau)\phi(S_0, S, \tau^-)dS \\ &+ e^{-r\tau} \int_{S^*}^{\infty} (S - K)\phi(S_0, S, \tau^-)dS \end{aligned} \tag{4}$$

where, $C_A(S, 0; D_\tau, \tau)$ denotes the price of an American call option with strike price *K* and maturity *T* at time 0 when the current stock price is S and the dividend at τ is D_τ. In addition, $C_E(S, t)$ denotes the price of a European call option at time *t* when the stock price is S and there is no dividend until the maturity T. Additionally, $\phi(S_0, S, \tau^-)$ denotes the risk neutral transition probability density function from the price S_0 at time 0 to the price S at time τ^-.

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[6] Etoré and Gobet (2012) investigate European options under these assumptions.

[7] When early exercise is not optimal, the value is equal to the intrinsic value, otherwise, it is equal to the price of a European option at τ.

Note that the interval of the integral is $\left(\frac{\delta}{1-y},\infty\right)$ because S_τ becomes zero when $S_{\tau^-}=\frac{\delta}{1-y}$. Finally, S^* denotes the lowest stock price for an early exercise and satisfies the following condition:[8]

$$S^* - K = C_E(S^*(1-y)-\delta,\tau) \tag{5}$$

3. Analytical approximations of option prices

Via substitution into the Black and Scholes formula, we replace Equation (4) with the following equation:

$$\begin{aligned} C_A(S_0,0;D_\tau,\tau) = e^{-r\tau}\int_{\frac{\delta}{1-y}}^{S^*} &\left((S(1-y)-\delta)N\left(\frac{\ln(S(1-y)-\delta)+c_+}{\sigma\sqrt{T-\tau}}\right)\right. \\ &\left.-Ke^{-r(T-\tau)}N\left(\frac{\ln(S(1-y)-\delta)+c_-}{\sigma\sqrt{T-\tau}}\right)\right)\phi(S_0,S,\tau^-)dS \\ &+S_0N(d_+)-Ke^{-r\tau}N(d_-) \end{aligned} \tag{6}$$

for some constants c_+, c_-, d_+, and d_-, as well as the cumulative standard normal distribution function *N*. When $\delta\neq 0$, the terms

$$\frac{\ln(S(1-y)-\delta)+c_\pm}{\sigma\sqrt{T-\tau}} \tag{7}$$

prevent a simplification of the integration because these do not follow a well-known distribution. Therefore, the main approximation strategy is to simplify them. For convenience, let us define $\bar{S}$ as follows:

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[8] When $y=0$ and $\delta<K(1-e^{-rT})$, S^* is infinity because Equation (5) has no solution. In addition, Equation (4) becomes the price of a European call option when we use $S^*=\infty$, regardless of Equation (5).

$$\bar{S} \equiv S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)\tau} \tag{8}$$

Using this notation, the interim stock prices are

$$S_{\tau^-} = \bar{S}e^{\sigma\sqrt{\tau}z} \tag{9}$$

$$S_\tau = \bar{S}e^{\sigma\sqrt{\tau}z}(1-y) - \delta \tag{10}$$

where z represents a random variable that follows a standard normal distribution. Then, by the Taylor expansion, we can approximate $\ln S_\tau$ around $\bar{z}$ as follows:[9]

$$\ln S_\tau \approx a_{o,1} + a_{o,2}(z-\bar{z}) + a_{o,3}(z-\bar{z})^2 \tag{11}$$

where

$$a_{o,1} = \ln\left(\bar{S}e^{\sigma\sqrt{\tau}\bar{z}}(1-y) - \delta\right) \tag{12}$$

$$a_{o,2} = \frac{\sigma\sqrt{\tau}\bar{S}e^{\sigma\sqrt{\tau}\bar{z}}(1-y)}{\bar{S}e^{\sigma\sqrt{\tau}\bar{z}}(1-y) - \delta} \tag{13}$$

$$a_{o,3} = -\frac{\delta\sigma^2\tau\bar{S}e^{\sigma\sqrt{\tau}\bar{z}}(1-y)}{2\left(\bar{S}e^{\sigma\sqrt{\tau}\bar{z}}(1-y) - \delta\right)^2} \tag{14}$$

Note that when we use the first order approximation with $a_{o,3} = 0$, the log stock price and Equation (7) at τ follows a normal distribution. Therefore, we can obtain an analytical solution for Equation (6) under this approximation. Furthermore, when $\delta = 0$, the solution is the exact value rather than an approximation because Equation (11) holds exactly.

[Figure 1 about here]

Figure 1 compares the accuracy of approximations of option value $C_E\left((1-y)S_{\tau^-} - \delta, \tau\right)$ at time τ^- as a function of S_{τ^-}. Besides the exact option value, the figure includes option values under quadratic approximations of $\ln S_\tau$ (log quadratic approximation), under the linear approximation of $\ln S_\tau$ (log linear approximation), and under the escrow model (escrow

[9] For convenience, we denote a_i instead of $a_i(\bar{z})$, although a_i for all i is a function of $\bar{z}$.

approximation), as well as the linear approximation of the option price (price approximation). In Panel A, all curves except the price approximation are difficult to distinguish. Hence, Panel B describes the absolute error of each approximation and shows that the log linear approximation has a smaller error than the escrow approximation. With this property, we use the log linear approximation to obtain the approximation presented in Proposition 1.

Proposition 1. For the upper bound of the American call option price:

we define $\overline{C}_A(\overline{z})$ as follows:

$$\begin{aligned}\overline{C}_A(\overline{z}) &= e^{-r\tau}\left(F_1(z_{max}, a_l, b_l; \overline{z}) - F_1(z_{min}, a_l, b_l; \overline{z})\right)\\ &-Ke^{-rT}\left(F_1(z_{max}, 0, c_l; \overline{z}) - F_1(z_{min}, 0, c_l; \overline{z})\right)\\ &+S_0N(d_+) - Ke^{-r\tau}N(d_-)\end{aligned} \tag{15}$$

where

$$F_1(z, a, b; \overline{z}) = e^{a_1+\frac{a_2^2}{2}}N_2\left(\left(z - a_2, \frac{b_1 + a_2 b_2}{\sqrt{1+b_2^2}}\right)', -\frac{b_2}{\sqrt{1+b_2^2}}\right) \tag{16}$$

and

$$z_{min} = \frac{\ln(\delta/(1-y)) - \ln(\bar{S})}{\sigma\sqrt{\tau}} \tag{17}$$

$$z_{max} = \frac{\ln(S^*) - \ln(\bar{S})}{\sigma\sqrt{\tau}} \tag{18}$$

$$a_{l,1} = a_{o,1} - \overline{z}a_{o,2} \tag{19}$$

$$a_{l,2} = a_{o,2} \tag{20}$$

$$b_{l,1} = \frac{a_{l,1} - \ln K + \left(r + \frac{1}{2}\sigma^2\right)(T-\tau)}{\sigma\sqrt{T-\tau}} \tag{21}$$

$$c_{l,1} = b_{l,1} - \sigma\sqrt{T-\tau} \tag{22}$$

$$b_{l,2} = c_{l,2} = \frac{a_{l,2}}{\sigma\sqrt{T-\tau}} \tag{23}$$

$$d_- = -z_{max} \tag{24}$$

$$d_+ = d_- + \sigma\sqrt{\tau} \tag{25}$$

Then, $\overline{C}_A(\overline{z})$ satisfies the following properties:

1) $\overline{C}_A(\overline{z})$ is an upper bound of the price of the American call option.

2) When $\delta = 0$, $\overline{C}_A(\overline{z})$ is the price of the American call option.

The proof is given in the Appendix.

Proposition 1 states that $\overline{C}_A(\overline{z})$ is not only an approximation but also an upper bound of the American call option price. To obtain the formula in Proposition 1, we rely on the small error of the log linear approximation and approximate all interim prices $S(1-y)-\delta$ in Equation (6) as log linear. However, as mentioned above, the intractability of Equation (6) arises from the interim prices inside the function *N*. Hence, we can find another formula by approximating interim prices only inside the function *N*, as presented in Proposition 2.

Proposition 2. For the lower bound of the American call option price:

we define $\underline{C_A}(\overline{z})$ as:

$$\begin{aligned} \underline{C_A}(\overline{z}) = {} & e^{-r\tau}\left(F_1(z_{max}, a_e, b_l; \overline{z}) - F_1(z_{min}, a_e, b_l; \overline{z})\right) \\ & -\delta e^{-r\tau}\left(F_1(z_{max}, 0, b_l; \overline{z}) - F_1(z_{min}, 0, b_l; \overline{z})\right) \\ & -Ke^{-rT}\left(F_1(z_{max}, 0, c_l; \overline{z}) - F_1(z_{min}, 0, c_l; \overline{z})\right) \\ & +S_0N(d_+) - Ke^{-r\tau}N(d_-) \end{aligned} \tag{26}$$

where

$$a_{e,1} = \ln\left(\bar{S}(1-y)\right) \tag{27}$$

$$a_{e,2} = \sigma\sqrt{\tau} \tag{28}$$

Then, $\underline{C_A}(\overline{z})$ satisfies the following properties:

1) $\underline{C_A}(\overline{z})$ is a lower bound of the price of the American call option.

2) When $\delta = 0$, $\underline{C_A}(\overline{z})$ is the price of the American call option.

The proof is given in the Appendix.

As Proposition 1 provides an upper bound, Proposition 2 states that our second approximation $\underline{C_A}(\overline{z})$ is a lower bound. Since the stock price outside function *N* is not replaced by the approximation to derive $\underline{C_A}(\overline{z})$, we can presume that $\underline{C_A}(\overline{z})$ has smaller error than $\overline{C}_A(\overline{z})$, though $\underline{C_A}(\overline{z})$ has more terms than $\overline{C}_A(\overline{z})$ does.

There are some practical issues related to obtaining these two approximations. First, $\overline{C}_A(\overline{z})$ and $\underline{C_A}(\overline{z})$ require the value S^*, which is the minimum stock price for the early exercise. Because Equation (5) has no explicit expression, we require a numerical method. However, Ju's (1998) finding from a study of American put options indicates that the error related to the boundary of early exercise is not critical to the accuracy of our solution. Therefore, a few iterations of Newton's method from the original strike price *K* is enough for the approximation. The other issue is how to improve the accuracy. For this, note that we conduct a (log) linear approximation around $\overline{z}$, which is an arbitrary number between z_{min} and z_{max}. We can thus improve the accuracy by choosing an optimal $\overline{z}$ or by using a set of $\overline{z}_i$ with subintervals within the interval (z_{min}, z_{max}). Using the latter method, we suggest more accurate upper and lower bounds of the American call option price as follows:

Corollary 3. For the upper and lower bounds of call option prices:

we define $\overline{C}_{A,Z,\bar{Z}}$ and $\underline{C}_{A,Z,\bar{Z}}$ as follows:

$$\begin{aligned}\overline{C}_{A,Z,\bar{Z}} &= e^{-r\tau}\sum_{i=1}^{n}\left(F_1(z_i,a_l,b_l;\bar{z}_i) - F_1(z_{i-1},a_l,b_l;\bar{z}_i)\right) \\ &-Ke^{-rT}\sum_{i=1}^{n}\left(\left(F_1(z_i,0,c_l;\bar{z}_i) - F_1(z_{i-1},0,c_l;\bar{z}_i)\right)\right) \\ &+S_0N(d_+) - Ke^{-r\tau}N(d_-)\end{aligned} \tag{29}$$

$$\begin{aligned}\underline{C}_{A,Z,\bar{Z}} &= e^{-r\tau}\sum_{i=1}^{n}\left(F_1(z_i,a_e,b_l;\bar{z}_i) - F_1(z_{i-1},a_e,b_l;\bar{z}_i)\right) \\ &-\delta e^{-r\tau}\sum_{i=1}^{n}\left(\left(F_1(z_i,0,b_l;\bar{z}_i) - F_1(z_{i-1},0,b_l;\bar{z}_i)\right)\right) \\ &-Ke^{-rT}\sum_{i=1}^{n}\left(\left(F_1(z_i,0,c_l;\bar{z}_i) - F_1(z_{i-1},0,c_l;\bar{z}_i)\right)\right) \\ &+S_0N(d_+) - Ke^{-r\tau}N(d_-)\end{aligned} \tag{30}$$

where Z is a set $\{z_0, \dots, z_n\}$ such that $z_{min} = z_0 < z_1 < \cdots < z_n = z_{max}$ and $\bar{Z}$ is a set $\{\bar{z}_1, \dots, \bar{z}_n\}$ such that $z_{i-1} < \bar{z}_i < z_i$. Then,

1) $\overline{C}_{A,Z,\bar{Z}}$ and $\underline{C}_{A,Z,\bar{Z}}$ are the upper and lower bounds of the American call option, respectively.

2) When $\delta = 0$, $\overline{C}_{A,Z,\bar{Z}}$ and $\underline{C}_{A,Z,\bar{Z}}$ are the prices of the American call option.

3) $\overline{C}_{A,Z,\bar{Z}}$ and $\underline{C}_{A,Z,\bar{Z}}$ converge to the exact value as the partition becomes finer.

Corollary 3 states that adopting finer partitions can improve accuracy of the prices, it can significantly increase computing time. Therefore, let us consider the other approach, the (log) quadratic approximation. As Figure 1 shows, the quadratic approximation has the lowest error among the approximations, and thus the following approximations can improve the accuracy.

$$e^{-r\tau}\int_{z_{min}}^{z_{max}}\Big(e^{a_1+a_2z+a_3z^2}N(b_1+b_2z+b_3z^2)$$
$$-Ke^{-r(T-\tau)}N(c_1+c_2z+c_3z^2)\Big)n(z)dz+S_0N(d_+)$$
$$-Ke^{-r\tau}N(d_-) \tag{31}$$

$$e^{-r\tau}\int_{z_{min}}^{z_{max}}\Big((\bar{S}e^{\sigma\sqrt{\tau}z}(1-\alpha y)-\alpha\delta)N(b_1+b_2z+b_3z^2)$$
$$-Ke^{-r(T-\tau)}N(c_1+c_2z+c_3z^2)\Big)n(z)dz+S_0N(d_+)$$
$$-Ke^{-r\tau}N(d_-) \tag{32}$$

However, as opposed to the linear approximation, Equations (31) and (32) are not in analytical forms due to the terms b_3 and c_3 in the expressions. Here we adopt Li *et al.*'s (2008) approach, which approximates the cumulative distribution function via second order Taylor expansion around $b_3x^2=0$, as follows:

$$\int e^{a_1+a_2x+a_3x^2}N(b_1+b_2x+b_3x^2)n(x)dx$$
$$\approx\int e^{a_1+a_2x+a_3x^2}n(x)N(b_1+b_2x)dx$$
$$+b_3\int x^2e^{a_1+a_2x+a_3x^2}n(x)n(b_1+b_2x)dx$$
$$-\frac{1}{2}b_3^2\int x^4(b_1+b_2x)e^{a_1+a_2x+a_3x^2}n(x)N(b_1+b_2x)dx \tag{33}$$

Then, Proposition 4 simplifies Equations (31) and (32) according to Equation (33).

Proposition 4. The quadratic approximations of call option prices

We simplify Equations (31) and (32) to Equations (34) and (35), respectively:[10]

[10] Although Equation (16) is a special case of Equation (37) with $a_3=0$, we use the same notation F_1 for convenience.

$$C_A^{ful}(\overline{z}) = F\left(z_{max}, a_q, b_q; \overline{z}\right) - F\left(z_{min}, a_q, b_q; \overline{z}\right)$$
$$-Ke^{-rT}\left(F\left(z_{max}, 0, c_q; \overline{z}\right) - F\left(z_{min}, 0, c_q; \overline{z}\right)\right) \tag{34}$$
$$+S_0 N(d_+) - Ke^{-r\tau} N(d_-)$$

$$C_A^{part}(\overline{z}) = F\left(z_{max}, a_e, b_q; \overline{z}\right) - F\left(z_{min}, a_e, b_q; \overline{z}\right)$$
$$-\delta e^{-r\tau}\left(F\left(z_{max}, 0, b_q; \overline{z}\right) - F\left(z_{min}, 0, b_q; \overline{z}\right)\right)$$
$$-Ke^{-rT}\left(F\left(z_{max}, 0, c_q; \overline{z}\right) - F\left(z_{min}, 0, c_q; \overline{z}\right)\right) \tag{35}$$
$$+S_0 N(d_+) - Ke^{-r\tau} N(d_-)$$

where

$$F(x, a, b; \overline{z}) \equiv F_1 + F_2 \exp\left(a_1 - \frac{b_1^2 + 2j_2 x + j_3^2 x^2}{2}\right)$$
$$+F_3 \exp\left(a_1 + \frac{j_2^2}{2j_3^2} - \frac{b_1^2}{2}\right) N\left(j_3 x + \frac{j_2}{j_3}\right) \tag{36}$$

$$F_1 = j_1 e^{a_1 + \frac{a_2^2 j_1^2}{2}} N_2\left(\left(\frac{x - j_1^2 a_2}{j_1}, \frac{b_1 + a_2 b_2 j_1^2}{\sqrt{1 + b_2^2 j_1^2}}\right)', -\frac{b_2 j_1}{\sqrt{1 + b_2^2 j_1^2}}\right) \tag{37}$$

$$F_2 = \frac{j_2 - j_3^2 x}{2\pi j_3^4} b_3 + \frac{j_4 + 3j_3^4 x - 5j_2 j_3^2}{4\pi j_3^8} b_1 b_3^2$$
$$+\frac{j_3^8 x^4 - j_2 j_4 + 4j_3^6 x^2 - 7j_2 j_3^4 x + 9j_2^2 j_3^2 + 8j_3^4}{4\pi j_3^{10}} b_2 b_3^2 \tag{38}$$

$$F_3 = \frac{j_3^2 + j_2^2}{\sqrt{2\pi} j_3^5} b_3 - \frac{3j_3^4 + 6j_2^2 j_3^2 + j_2^4}{2\sqrt{2\pi} j_3^9} b_1 b_3^2 + \frac{15j_2 j_3^4 + 10j_2^3 j_3^2 + j_2^5}{2\sqrt{2\pi} j_3^{11}} b_2 b_3^2 \tag{39}$$

with

$$a_{q,1} = a_{o,1} - 2\bar{\mathrm{z}}a_{o,2} + \bar{\mathrm{z}}^2 a_{o,3} \tag{40}$$

$$a_{q,2} = a_{o,2} - 2\bar{\mathrm{z}}a_{o,3} \tag{41}$$

$$a_{q,3} = a_{o,3} \tag{42}$$

$$b_{q,1} = \frac{a_{q,1} - \ln K + \left(r + \frac{1}{2}\sigma^2\right)(T - \tau)}{\sigma\sqrt{T - \tau}} \tag{43}$$

$$c_{q,1} = b_{q,1} - \sigma\sqrt{T - \tau} \tag{44}$$

$$b_{q,2} = c_{q,2} = \frac{a_{q,2}}{\sigma\sqrt{T - \tau}} \tag{45}$$

$$b_{q,3} = c_{q,3} = \frac{a_{q,3}}{\sigma\sqrt{T - \tau}} \tag{46}$$

$$j_1 = \frac{1}{\sqrt{1 - 2a_3}} \tag{47}$$

$$j_2 = b_1 b_2 - a_2 \tag{48}$$

$$j_3 = \sqrt{1 + b_2^2 - 2a_3} \tag{49}$$

$$j_4 = (j_3^2 x - j_2)(j_3^4 x^2 + j_2^2) \tag{50}$$

The proof is given in the Appendix.

4. Performance Comparison

In this section, we compare the performance of the formulas presented in the previous section. For each method, we measure the calculation speed in computing time (seconds) for 10,000 options and the accuracy of jth method measured by the relative root mean square error (RRMSE), which is

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{C_{j,i}-C_{b,i}}{C_{b,i}}\right)^2}, \tag{51}$$

the root mean square error (RMSE), which is

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(C_{j,i}-C_{b,i}\right)^2}, \tag{52}$$

and the maximum absolute error (MAE), which is

$$\max_{i}\left|C_{j,i}-C_{b,i}\right|, \tag{53}$$

where $C_{j,i}$ represents the value of the *i*th option using the *j*th method and $C_{b,i}$ represents the benchmark value.[11] For options benchmark values, we use the lower bound in Corollary 3 with a fine partition.[12]

In Table 1, we denote the pricing formulas in Propositions 1 and 2 as Upper and Lower, respectively; and the pricing formulas in Proposition 4 as Full and Part. To obtain the option values with these methods, we set $\bar{z}=0$ and use Newton's method with an initial guess of exercise price *K* and 4 iterations to obtain S^*. For comparison, we also present the AR(*x*) result from Areal and Rodrigues' (2013) method with 1000*x* nodes because their method is the most efficient, to the best of our knowledge.

[Table 1 around here]

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[11] Options with value less than 0.5 are not included in the RRMSE calculation.

[12] More specifically, we take set Z $(=\{z_0,\dots,z_n\})$ as a subset of $\{z_{min}, z_{max}, -10, -6, -5.9, \dots, -4.1, -4, -3.95, \dots, -2.05, -2, -1.99, \dots, 2, 2.05, \dots, 4, 4.1, \dots, 6, 10\}$ within the interval $[z_{min}, z_{max}]$ and set $\bar{z}_i=\frac{z_{i-1}+3z_i}{4}$. In addition, for an accurate S^*, we use Newton's method with an initial guess of exercise price *K* and 20 iterations. In a similar manner, we obtain an upper bound from Corollary 3. The RRMSE between these two bounds is 6.5×10^{-8} and it shows the accuracy of the benchmark value.

Table 1 compares the speed and accuracy of the pricing formulas. It shows that the Upper and Full are the least accurate methods, though they are slower than Areal and Rodrigues (2013) method with 1000 nodes. However, it is more accurate to approximate the stock price only inside the cumulative distribution functions. As a result, both Lower and Part become more efficient. Specifically, Lower is better than AR(3) because it is more accurate and less time-consuming. Similarly, Part is better than AR(5) and AR(10).

[Table 2 and 3 around here]

While Table 1 shows only the average accuracy of the formulas, Tables 2 and 3 show the accuracy for individual cases. According to these tables, the errors (absolute deviation) for Lower and Part increase when the underlying asset is volatile and the dividend timing is about half of the time to maturity. For example, the maximum error of Lower in Table 2 is 0.6878 when $\tau = 0.5$, $\sigma = 50\%$, and K=120; and the maximum error of Lower in Table 3 is 8.557 when $\tau = 0.5$, $\sigma = 50\%$, and K=80. Nevertheless, both tables show that the errors of Part are always smaller than the errors of AR(10). In addition, in Table 2, the errors of Lower are smaller than those of AR(10), except in two cases. In Table 3, although the maximum error of Lower is greater than that of AR(10), it is smaller than that of AR(5). The errors of Lower are smaller than those of AR(5), except in seven cases. Thus, Part is the most accurate of the methods, and the accuracy of Lower is comparable to AR(5), although the errors (absolute deviation) of Lower and Part increase when the timing of the dividend is midway to the time to maturity and the underlying asset is volatile.

5. Conclusion

For exotic options pricing, there are frequent studies into analytical approximations when the options have no analytical closed form solution, likely because of the efficiency and accuracy of the approximation method. In this regard, analytical approximations can improve the speed and accuracy of pricing American options with discrete dividends.

In this paper, we provide analytical approximation formulas for American call option prices when there is a discrete affine dividend. Our approximations include the lower and upper bounds of option prices. Moreover, numerical analysis shows that our lower bound pricing formula and quadratic partial approximation formula are very efficient. We hope this study triggers research into pricing American options with discrete dividends and future improvements.

Appendix

Lemma 1.

$$A_1(x,a,b) \equiv \int_{-\infty}^{x} e^{a_1+a_2z+a_3z^2} n(z)N(b_1+b_2z)dz$$
$$= g_1 e^{a_1+\frac{a_2^2g_1^2}{2}} N_2\left(\left(\frac{x-g_1^2a_2}{g_1}, \frac{b_1+a_2b_2g_1^2}{\sqrt{1+b_2^2g_1^2}}\right)', -\frac{b_2g_1}{\sqrt{1+b_2^2g_1^2}}\right) \quad \text{(A1)}$$

where $g_1 = 1/\sqrt{1-2a_3}$ and $N_2(.,\rho)$ is a bivariate normal cumulative distribution function with zero mean and covariance matrix $\begin{pmatrix}1 & \rho \\ \rho & 1\end{pmatrix}$. In particular, when $a_3 = 0$, it becomes:

$$\int_{-\infty}^{x} e^{a_1+a_2z} n(z)N(b_1+b_2z)dz = e^{a_1+\frac{a_2^2}{2}} N_2\left(\left(x-a_2, \frac{b_1+a_2b_2}{\sqrt{1+b_2^2}}\right)', -\frac{b_2}{\sqrt{1+b_2^2}}\right) \quad \text{(A2)}$$

Proof:

According to Geske (1979b), we have the following relationship:

$$N_2((h,k)',\rho) = \int_{-\infty}^{h} n(x)N\left(\frac{k-\rho x}{\sqrt{1-\rho^2}}\right)dx \quad \text{(A3)}$$

Therefore, by adopting the following equation, we get the identity of Lemma 1.

$$e^{a_1+a_2z+a_3z^2} n(z) = e^{a_1+\frac{a_2^2g_1^2}{2}} n\left(\frac{z-g_1^2a_2}{g_1}\right) \quad \text{(A4)}$$

■

Lemma 2.

Let us define functions $f(S)$ and $d_\pm(S)$ as follows:

$$f(S) = S_0e^{-qT}N(d_+(S)) - Ke^{-rT}N(d_-(S)) \quad \text{(A5)}$$

$$d_\pm(S) = \frac{\ln S - \ln K + \left(r - q \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \text{(A6)}$$

Then, the function $f(S)$ is maximized when $\mathrm{S} = \mathrm{S}_0$

Proof:

Let x be $(\ln S - \ln S_0)/\sigma\sqrt{T}$. Then $h(x)$ represents $f(S)$, as follows:

$$h(x) = S_0 e^{-qT} N(d_+(S_0) + x) - K e^{-rT} N(d_-(S_0) + x) \tag{A7}$$

Because $\ln(\cdot)$ is an increasing function, it is enough to show that $h(x)$ is maximized when $x = 0$. Meanwhile, differentiation of h yields:

$$h'(x) = \sqrt{\frac{S_0 K}{2\pi}} e^{\left(\frac{(-r-q)T}{2} - \frac{\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r-q)T}{\sigma\sqrt{T}} + x\right)^2}{2} - \frac{\sigma^2 T}{8}\right)} \left(exp\left(-\frac{\sigma\sqrt{T}x}{2}\right) - exp\left(\frac{\sigma\sqrt{T}x}{2}\right)\right) \tag{A8}$$

This implies that $h'(\mathrm{x}) < 0$ if $x > 0$ and $h'(x) > 0$ if $x < 0$. Therefore, $h(x)$ is maximized when $x = 0$.

■

Lemma 3. Useful integration

$$\begin{aligned} &\int e^{a_1 + a_2 x + a_3 x^2} n(x) n(b_1 + b_2 x) x^2 dx \\ &= \frac{j_2 - j_3^2 x}{2\pi j_3^4} e^{a_1 - \frac{b_1^2 + 2 j_2 x + j_3^2 x^2}{2}} \\ &+ \frac{j_2^2 + j_3^2}{\sqrt{2\pi} j_3^5} e^{a_1 + \frac{B^2}{2C^2} - \frac{b^2}{2}} N\left(j_3 x + \frac{j_2}{j_3}\right) \end{aligned} \tag{A9}$$

$$\begin{aligned} &\int e^{a_1 + a_2 x + a_3 x^2} n(x) n(b_1 + b_2 x) x^4 dx \\ &= -\frac{j_4 + 3 j_3^4 x - 5 j_2 j_3^2}{2\pi j_3^8} e^{a_1 - \frac{b_1^2 + 2 j_2 x + j_3^2 x^2}{2}} \\ &+ \frac{3 j_3^4 + 6 j_2^2 j_3^2 + j_2^4}{\sqrt{2\pi} j_3^9} e^{a_1 + \frac{B^2}{2C^2} - \frac{b^2}{2}} N\left(j_3 x + \frac{j_2}{j_3}\right) \end{aligned} \tag{A10}$$

$$\int e^{a_1+a_2x+a_3x^2} n(x)n(b_1+b_2x)x^5 dx$$

$$= -\frac{j_3^8x^4 - j_2j_4 + 4j_3^6x^2 - 7j_2j_3^4x + 9j_2^2j_3^2 + 8j_3^4}{2\pi j_3^{10}} e^{a_1-\frac{b_1^2+2j_2x+j_3^2x^2}{2}} \tag{A11}$$

$$-\frac{15j_2j_3^4 + 10j_2^3j_3^2 + j_2^5}{\sqrt{2\pi}j_3^{11}} e^{a_1+\frac{B^2}{2C^2}-\frac{b^2}{2}} N\left(j_3x+\frac{j_2}{j_3}\right)$$

Therefore, we have the following approximation:

$$\int e^{a_1+a_2x+a_3x^2} n(x)N(b_1+b_2x+b_3x^2)dx$$

$$\approx \int e^{a_1+a_2x+a_3x^2} n(x)N(b_1+b_2x)dx$$

$$+ h_2 \int x^2 e^{a_1+a_2x+a_3x^2} n(x)n(b_1+b_2x)dx$$

$$-\frac{1}{2}h_2^2 \int x^4(h_0+h_1x)e^{a_1+a_2x+a_3x^2} n(x)N(b_1+b_2x)dx \tag{A12}$$

$$= F_1(x,a,b) + F_2(x,a,b)\exp\left(a_1 - \frac{b_1^2+2j_2x+j_3^2x^2}{2}\right)$$

$$+ F_3(x,a,b)\exp\left(a_1+\frac{j_2^2}{2j_3^2}-\frac{h_0^2}{2}\right) N\left(j_3x+\frac{j_2}{j_3}\right)$$

where

$$j_1 = \frac{1}{\sqrt{1-2a_3}}, \tag{A13}$$

$$j_2 = b_1b_2 - a_2, \tag{A14}$$

$$j_3 = \sqrt{1+b_2^2-2a_3}, \tag{A15}$$

$$j_4 = (j_3^2 x - j_2)(j_3^4 x^2 + j_2^2) \tag{A16}$$

$$F_1 = j_1 e^{a_1 + \frac{a_2^2 j_1^2}{2}} N_2 \left(\left(\frac{x - j_1^2 a_2}{j_1}, \frac{b_1 + a_2 b_2 j_1^2}{\sqrt{1 + b_2^2 j_1^2}} \right)', -\frac{b_2 j_1}{\sqrt{1 + b_2^2 j_1^2}} \right) \tag{A17}$$

$$\begin{aligned} F_2 = {} & \frac{j_2 - j_3^2 x}{2\pi j_3^4} b_3 + \frac{j_4 + 3 j_3^4 x - 5 j_2 j_3^2}{4\pi j_3^8} b_1 b_3^2 \\ & + \frac{j_3^8 x^4 - j_2 j_4 + 4 j_3^6 x^2 - 7 j_2 j_3^4 x + 9 j_2^2 j_3^2 + 8 j_3^4}{4\pi j_3^{10}} b_2 b_3^2 \end{aligned} \tag{A18}$$

$$F_3 = \frac{j_3^2 + j_2^2}{\sqrt{2\pi} j_3^5} b_3 - \frac{3 j_3^4 + 6 j_2^2 j_3^2 + j_2^4}{2\sqrt{2\pi} j_3^9} b_1 b_3^2 + \frac{15 j_2 j_3^4 + 10 j_2^3 j_3^2 + j_2^5}{2\sqrt{2\pi} j_3^{11}} b_2 b_3^2 \tag{A19}$$

Proof of Proposition 1.

Substituting Equation (11) and $a_2 = 0$ in Equation (6) gives the following equation:

$$\begin{aligned} \overline{C}_A(\overline{z}) = e^{-r\tau} \int_{z_{min}}^{z_{max}} & \Big(e^{a_1 + a_2 z} N(b_1 + b_2 z) \\ & - K e^{-r(T-\tau)} N(c_1 + c_2 z) \Big) n(z) dz \\ & + S_0 N(d_+) - K e^{-r\tau} N(d_-) \end{aligned} \tag{A20}$$

Then by Lemma 1, we obtain Equation (15). Since the second order derivative a_2 is non-positive for all z, $\ln S_\tau$ is concave in z. Therefore, we have

$$\ln S_\tau \leq a_0 + a_1 (z - \overline{z}) \tag{A21}$$

Hence $\overline{C}_A(\overline{z})$ is an upper bound of the American call option price. In addition, when $\delta = 0$, Equation (11) becomes an equality. Therefore, $\overline{C}_A(\overline{z})$ becomes the exact American call option price.

■

Proof of Proposition 2.

Substituting Equation (11) and $a_2 = 0$ for the function N in Equation (6) yields the following equation:

$$\underline{C_A}(\overline{z}) = e^{-r\tau} \int_{z_{min}}^{z_{max}} \Big(\big(\bar{S} e^{\sigma\sqrt{\tau} z}(1-y) - \delta \big) N(b_1 + b_2 z) - K e^{-r(T-\tau)} N(c_1 + c_2 z) \Big) n(z) dz + S_0 N(d_+) - K e^{-r\tau} N(d_-) \tag{A22}$$

Then by Lemma 1, we obtain Equation (26). Additionally, from Lemma 2 we can show that $\underline{C_A}(\overline{z})$ is a lower bound. In addition, when $\delta = 0$, the approximation of Equation (11) becomes an equality. Therefore, $\underline{C_A}$ becomes the exact American call option price.

■

Proof of Proposition 4.

Lemma 1 yields the first part of $C_A^{ful}(\overline{z})$ and $C_A^{part}(\overline{z})$. In addition, Lemma 3 yields the second and third part of $C_A^{ful}(\overline{z})$ and $C_A^{part}(\overline{z})$.

■

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Table 1. Performance comparison

	Upper	Lower	Full	Part	AR(1)	AR(3)	AR(5)	AR(10)
Speed (sec)	2.87	3.51	3.95	4.79	2.50	4.82	6.65	11.02
RRMSE (bp)	37.89	0.41	4.89	0.03	2.29	0.70	0.40	0.20

This table compares the accuracy and speed of American option pricing methods when there is a single discrete dividend. The methods compared are those in Equation (15) in Proposition 1 (Upper), Equation (26) in Proposition 2 (Lower), Equation (34) in Proposition 4 (Full), and Equation (35) in Proposition 4 (Part). We also present the results from Areal and Rodrigues (2013) because their method is very efficient; results with 1000*x* nodes are denoted by AR(*x*). For each method, the calculation speed is measured by computing time (seconds) for 10,000 options and accuracy is measured by the RRMSE, represented in Equation (51), in a basis point. For the analysis, we generate 10,000 sets of parameters such that $K = 100$, $\alpha = 1$, $S \sim U(70{,}130)$, $\sigma \sim U(0.1{,}0.6)$, $r \sim U(0{,}0.01)$, $\delta \sim U(1{,}10)$, $y \sim U(0{,}0.05)$, $T \sim u(\frac{2}{360}, 1)$, and $\tau \sim u(\frac{1}{360}, T - \frac{1}{360})$ with the restriction $D > K(1 - e^{-rT})$ where $X \sim U(a,b)$ and $X \sim u(a,b)$ indicate that X is generated from the uniform distribution in a real interval (a,b) and a range of $\left\{a, a + \frac{1}{360}, a + \frac{2}{360}, \dots, b\right\}$, respectively. The experiment is conducted using Matlab 2015a and code for the cumulative bivariate normal distribution function from http://www.math.wsu.edu/faculty/genz.

Table 2. Performance comparison ($y = 1\%$ and $\delta = 2$)

σ	K	τ	Exact	Lower	Part	AR(5)	AR(10)
0.2	80	0.1	21.852	-0.002	0.000	0.020	0.010
0.2	80	0.5	22.555	-0.062	-0.001	0.110	-0.175
0.2	80	0.9	24.140	-0.084	-0.001	-0.428	0.087
0.2	90	0.1	14.355	-0.003	0.000	0.091	0.045
0.2	90	0.5	14.672	-0.039	0.000	0.615	0.348
0.2	90	0.9	16.176	-0.024	0.000	1.529	0.608
0.2	100	0.1	8.648	-0.004	0.000	0.098	0.049
0.2	100	0.5	8.789	-0.021	0.000	0.307	0.337
0.2	100	0.9	9.923	-0.002	0.000	2.022	0.226
0.2	110	0.1	4.800	-0.004	0.000	0.017	0.008
0.2	110	0.5	4.883	-0.030	-0.001	0.060	-0.014
0.2	110	0.9	5.590	-0.001	0.000	1.644	0.821
0.2	120	0.1	2.476	-0.004	0.000	-0.088	-0.044
0.2	120	0.5	2.532	-0.055	-0.001	-0.457	-0.226
0.2	120	0.9	2.917	-0.018	0.000	-0.520	0.034
0.5	80	0.1	29.497	-0.055	-0.002	0.218	0.109
0.5	80	0.5	29.864	-0.500	-0.029	1.166	0.873
0.5	80	0.9	31.197	-0.132	-0.004	-0.427	1.101
0.5	90	0.1	24.309	-0.060	-0.001	0.279	0.140
0.5	90	0.5	24.583	-0.408	-0.029	1.161	0.642
0.5	90	0.9	25.758	-0.037	-0.001	1.588	1.272
0.5	100	0.1	19.961	-0.063	-0.001	0.298	0.149
0.5	100	0.5	20.187	-0.408	-0.033	1.276	0.819
0.5	100	0.9	21.185	-0.014	0.000	3.458	1.950
0.5	110	0.1	16.354	-0.065	-0.001	0.278	0.139
0.5	110	0.5	16.552	-0.484	-0.038	1.025	0.551
0.5	110	0.9	17.382	-0.043	0.000	2.156	1.970
0.5	120	0.1	13.382	-0.066	-0.001	0.228	0.114
0.5	120	0.5	13.562	-0.608	-0.044	1.098	0.553
0.5	120	0.9	14.243	-0.151	-0.002	2.928	1.800
RMSE				0.205	0.014	1.226	0.769
MAE				0.608	0.044	3.458	1.970
RRMSE				0.012	0.001	0.102	0.052

This table shows the errors of pricing formulas for various parameters. We assume $S = 100$, $r = 5\%, T = 1,$ $y = 1\%$, and $\delta = 2$. The first three columns represent values of σ, K, and τ. The fourth column represents the benchmark values described in footnote 12. The fifth column describes the errors of the Lower method, which represent errors (Lower–benchmark) in basis points. The other columns similarly represent the errors of each method. The last three rows represent the RMSE, MAE, and RRMSE of each method according to Equations (51)-(53) in basis points.

Table 3. Performance comparison ($y = 5\%$ and $\delta = 10$)

σ	K	τ	Exact	Lower	Part	AR(5)	AR(10)
0.2	80	0.1	20.400	-0.006	0.000	-0.119	-0.072
0.2	80	0.5	22.188	-0.476	-0.022	-1.334	0.049
0.2	80	0.9	24.109	-0.011	-0.002	0.856	0.032
0.2	90	0.1	10.695	-0.060	0.000	2.306	0.436
0.2	90	0.5	13.570	-0.320	-0.010	2.200	1.357
0.2	90	0.9	16.099	-0.005	0.000	2.333	0.958
0.2	100	0.1	3.950	-0.039	0.000	-0.094	-0.635
0.2	100	0.5	7.061	-0.085	-0.002	4.356	2.200
0.2	100	0.9	9.799	0.000	0.000	-3.519	-1.627
0.2	110	0.1	1.536	-0.067	0.000	-2.038	-0.350
0.2	110	0.5	3.145	-0.031	0.000	0.851	-1.447
0.2	110	0.9	5.444	0.000	0.000	-1.648	-0.747
0.2	120	0.1	0.647	-0.081	-0.001	-0.220	-0.109
0.2	120	0.5	1.244	-0.244	-0.003	0.096	-0.211
0.2	120	0.9	2.783	-0.015	-0.001	0.339	0.483
0.5	80	0.1	22.920	-1.220	-0.029	11.289	4.510
0.5	80	0.5	27.228	-8.557	-0.728	0.420	3.076
0.5	80	0.9	30.841	-0.589	-0.041	-1.841	2.706
0.5	90	0.1	17.331	-1.123	-0.027	6.377	2.111
0.5	90	0.5	21.454	-5.233	-0.435	0.497	3.424
0.5	90	0.9	25.317	-0.118	-0.007	7.296	2.857
0.5	100	0.1	13.506	-1.318	-0.027	-1.361	-0.003
0.5	100	0.5	16.837	-3.212	-0.256	7.054	3.054
0.5	100	0.9	20.683	-0.022	-0.001	-4.839	4.177
0.5	110	0.1	10.700	-1.745	-0.032	-1.213	-0.649
0.5	110	0.5	13.214	-2.541	-0.156	-2.430	-1.560
0.5	110	0.9	16.843	-0.103	-0.001	-5.685	2.287
0.5	120	0.1	8.524	-1.999	-0.039	-0.126	-0.134
0.5	120	0.5	10.397	-3.664	-0.130	1.760	1.947
0.5	120	0.9	13.690	-0.702	-0.020	4.637	0.950
RMSE				2.188	0.167	3.749	1.957
MAE				8.557	0.728	11.289	4.510
RRMSE				0.131	0.008	0.350	0.163

This table is similar to Table 2, except that we assume $y = 5\%$ and $\delta = 10$.

Figure 1. Option value and stock price immediately before the ex-dividend date.

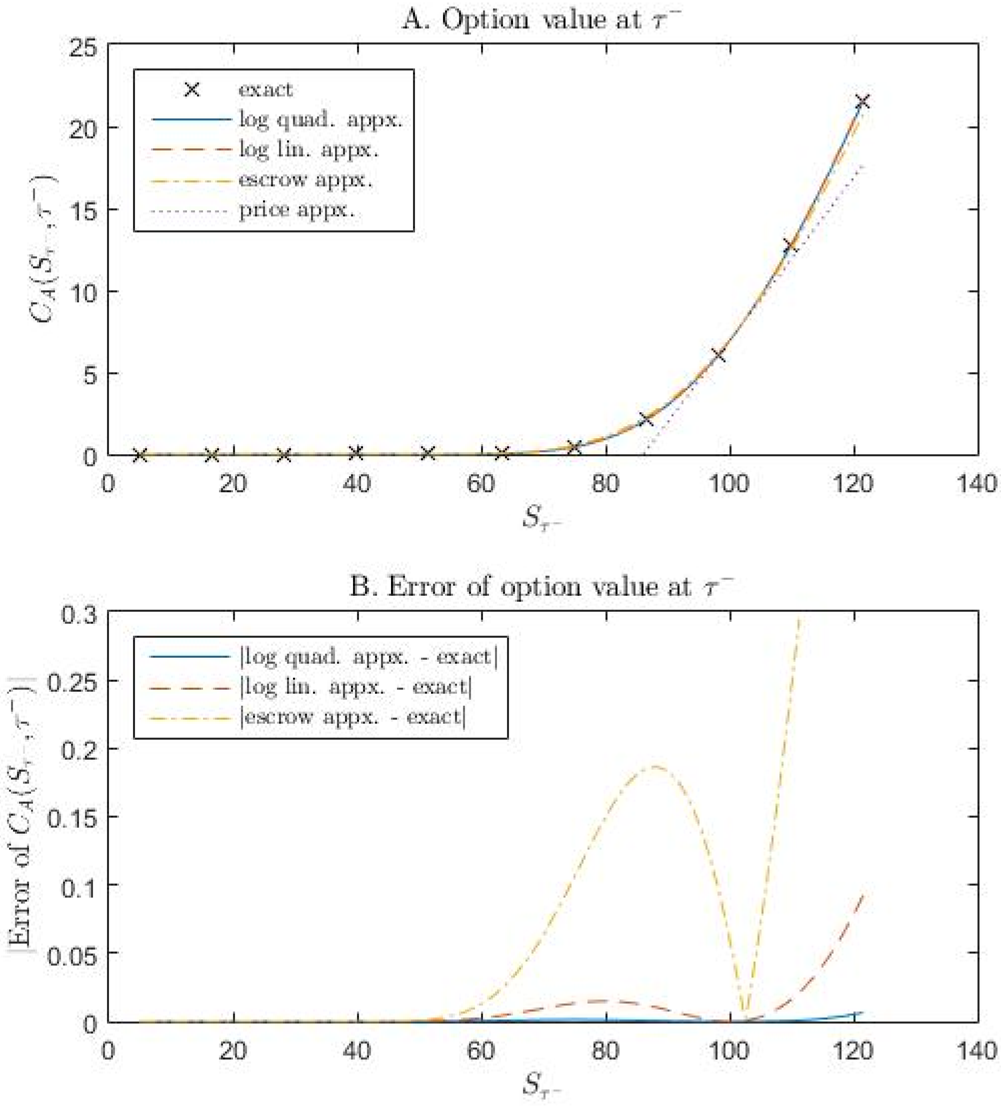

Figure 1 compares the accuracy of approximations for $C_A(S_{\tau^-}, \tau^-)$, which is the option value just before the ex-dividend date, for the no-early-exercise region. Because early-exercise is not optimal in the domain, $C_A(S_{\tau^-}, \tau^-)$ is equivalent to $C_E(S_{\tau^-}, \tau^-)$ and $C_E(S_{\tau^-}(1-y)-\delta, \tau)$, which appears in Equation (4). In the figure, 'exact' represents the exact value of an option. In addition, 'log quad. appx.' and 'log lin. appx.' represent log quadratic and log linear approximations, respectively. Additionally, 'escrow appx.' represents option value in the escrow model, such as those in Roll (1977), Geske (1979a), Whaley (1981). 'price appx.'

represents approximation with a tangent line such as in (de Matos et al. (2009)). We conduct the approximations around $\bar{S}\left(= S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)\tau}\right)$ or equivalently, $\bar{z} = 0$. The parameters for comparison are as follows: initial stock price $S_0 = 100$, exercise price $K = 100$, time to maturity $T = 1$, volatility $\sigma = 0.3$, risk free rate $r = 0.05$, dividend date $\tau = 0.5$, and dividend amount $\delta = 5$. With these parameters, the lowest price of early exercise is $121.5.