# Consumption, Retirement, and Asset Allocation with Unemployment Risks and Borrowing Constraints

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**Abstract**

We advance a model of consumption, retirement, and asset allocation in an incomplete market in which an individual is subject to risk of involuntary permanent unemployment that reduces her income severely, and has borrowing constraints. We show that the interactions among consumption and portfolio choice can induce early retirement even when forced unemployment risks and borrowing constraints are considered jointly. We demonstrate that providing private unemployment insurance in an incomplete market is beneficial to poor people and for people with a low post-retirement leisure preference, and that the insurance can be privately priced and be sold by private insurance providers.

keywords: optimal consumption, risky investment, retirement, risk of forced unemployment, borrowing constraints

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## 1 Introduction

Optimal life-cycle consumption and portfolio selection have received much academic attention in financial economics. Initial research in life-cycle models (Bodie *et al.*, 1992) examined the effect of the labor-leisure choice on an individual's optimal consumption and portfolio choice. Karatzas and Wang (2000) solved an optimal consumption and portfolio choice problem with discretionary stopping, without considering labor income. Building on these papers, Farhi and Panageas (2007) developed a theoretical model that can be used to examine interactions among consumption, risky investment, and retirement. Choi and Shim (2006) studied the optimal retirement and consumption and portfolio choice problem of an economic agent who obtains labor income but suffers a utility loss (disutility) from labor while working. Extending this work, Choi *et al.* (2008) solved the optimal retirement problem with consumption and leisure choice for the more general constant elasticity of substitution utility function than the constant relative risk aversion utility preference.

Among crucial elements that must be considered when studying consumption and portfolio choice over the life cycle, labor income risks have an important influence on an individual's optimal strategies (Heaton and Lucas, 1997; Koo, 1998; Viceira, 2001; Cocco *et al.*, 2005; Polkovnichenko, 2007; Benzoni *et al.*, 2007; Wachter and Yogo, 2010; Munk and Sørensen, 2010; Lynch and Tan, 2011). However, few researchers (Dybvig and Liu, 2010; Jang *et al.*, 2013, Bensoussan *et al.*, 2013) have addressed interactions among consumption, portfolio choice, and *voluntary retirement* with labor income risks.

Empirical or theoretical papers that explored the relationship between labor income risks and an individual's saving, asset composition, and retirement decision all seem to have a certain restrictive assumption, i.e., they do not consider an optimal choice of retirement time with labor income risks. To remedy this shortcoming, we aim to develop a theoretical model that includes unemployment risks in a utility-maximizing framework, and that can be used to derive important implications for the relationship between borrowing constraints and an individual's retirement behaviors. To the best of our knowledge, this is the first study to present a model that predicts the optimal choices of life-cycle consumption, portfolio, and retirement time for an individual who is subject to borrowing constraints and is exposed to an unhedgeable risk of forced unemployment.

More specifically, this paper makes three major contributions.

First, we advance a model of consumption, retirement, and asset allocation in an incomplete market; the model considers the case in which an individual is subject to risk of involuntary permanent unemployment that reduces her income severely,[1] and who has borrowing constraints. To focus how permanent unemployment shocks affect the individual's optimal strategies, the involuntary unemployment event is considered to be permanent; as a result, the risk of unemployment is equivalent to the risk of *forced* or *involuntary* retirement. Accordingly, an involuntarily unemployed person can be regarded as an involuntary retiree who is forced to retire when an unemployment shock arrives. Throughout the paper, we impose the borrowing constraints that an individual cannot borrow money with her unsecured or uncollateralized future income before retirement (either voluntary or involuntary).[2] Our model shows that individuals with the borrowing constraint significantly reduce consumption and risky investment even if the possibility of forced unemployment increases only slightly.

Second, we provide two useful concepts; "*implicit value of income*" (IVI) and "*certainty equivalent wealth gain*" (CEWG). We define IVI as the marginal rate of substitution between an individual's income and financial wealth. Then it is the individual's subjective marginal value of her labor, i.e., a criterion for the individual's optimal retirement decision; if IVI is higher than the implicit value of after-retirement income, the individual is willing to delay retirement; otherwise, she is willing to retire voluntarily. We also define CEWG as the largest wealth that the individual is willing to give up to eliminate the risk of unemployment; i.e., CEWG is compensation in return for bearing the risk of forced unemployment. To go into

[1]Social securities and private insurance market are insufficient to hedge perfectly against large and negative wealth shock (Gormley *et al.*, 2010). Furthermore, private insurance market in which an individual hedges against labor income risks is not competitive as compared to other insurance market (Cocco *et al.*, 2005). Therefore, we assume that social security insures some part of individual's labor income during periods of unemployment; i.e., that post-retirement income consists of unemployment allowances and income from other public welfare services. This assumption has been used in previous standard life-cycle models (Carroll *et al.*, 2003; Cocco *et al.*, 2005; Lynch and Tan, 2011).

[2]The wealth constraint has been considered in a complete market in the absence of labor income risks to investigate its impact on an individual's optimal strategies (Farhi and Panageas, 2007; Dybvig and Liu, 2010; Jang *et al.*, 2013). The crucial point here is that the borrowing constraint would be significant in an incomplete market, especially when risks to labor income exist. Most studies regarding life-cycle consumption and portfolio choice have emphasized that an individual cannot borrow money with her unsecured or uncollateralized future income. Several papers consider both labor income risk and borrowing constraint (Viceira, 2001; Cocco *et al.*, 2005; Polkovnichenko, 2007; Benzoni *et al.*, 2007; Wachter and Yogo, 2010; Munk and Sørensen, 2010).

significantly greater detail in terms of the economic analysis, we use these concepts and demonstrate how risk of forced unemployment affects optimal timing of retirement in the presence of borrowing constraints.

Finally, we suggest a market innovation in an incomplete market by introducing the private unemployment insurance proposed by Jang *et al.* (2013) and demonstrate that providing private unemployment insurance in an incomplete market is beneficial to poor people and for people with a low post-retirement leisure preference, and that the insurance can be privately priced and be sold by private insurance providers.[3] We consider an individual who is exposed to risk of forced unemployment, and define reservation purchase price (RPP) as the maximal lump-sum upfront premium that she is willing to pay for private unemployment insurance. We compute (RPP) of private unemployment insurance and can say that the insurance is marketable at an equivalent or lower price than the RPP if insurance companies can successfully eliminate the moral hazard problem of the policy holders. We also compute individual welfare benefit (IWB) of the market innovation. The IWB is defined as the maximum wealth that an individual is willing to give up to eliminate her risk of forced unemployment by purchasing the private unemployment insurance. Our model confirms a positive IWB, and shows that utility can be gained by introducing private unemployment insurance.

We acknowledge that the moral hazard problem is a major obstacle to adoption of private unemployment insurance. This is because if individuals privately observe forced unemployment, they are willing to retire earlier or even at an initial time. However, the model excludes this case by assuming that individuals cannot receive unemployment insurance coverage when they enter voluntary retirement. Actually, according to Jang *et al.* (2013) and related literature, insurance companies can minimize moral hazard problems by putting some provisions into insurance clauses.[4]

Using our model with carefully chosen parameters, the main results of this paper are

[3]Jang *et al.* (2013) considered a complete market in which an individual can purchase personalized unemployment insurance. They conjecture that there exists the loading factor of the unemployment insurance has an upper bound. An individual would be better off bearing all of the risk of involuntary unemployment rather than taking the insurance policy to hedge the risks if the loading is positive and very large.

[4]For future work, we can consider an insurance company that cannot distinguish whether or not the permanently unemployed are involuntarily retired people. This inability is very costly for the insurance company. Therefore, an interesting open problem is to consider an economy in which the insurance company cannot distinguish between voluntary retirement and involuntary retirement.

similar to results obtained in the literature on life-cycle consumption and portfolio with optimal retirement timing (Farhi and Panageas, 2007; Dybvig and Liu, 2010; Jang *et al.*, 2013; Bensoussan *et al.*, 2013). These papers provide some interesting observations that retirement flexibility reduces consumption and increases risky portfolio, and that borrowing constraints and labor income risks induce early retirement. What we show is that the interactions among consumption and portfolio choice can induce early retirement *even when forced unemployment risks and borrowing constraints are considered jointly.*

In this paper we extend a previous model (Bensoussan *et al.*, 2013) of optimal retirement in an incomplete market by adding a nonnegative wealth constraint. However, adding one more constraint to a retirement problem gives rise to unwanted complexity in solving the problem. A retirement problem with the nonnegative wealth constraint generally corresponds to a variational inequality with two free boundaries (Farhi and Panageas, 2007; Dybvig and Liu, 2010; Jang *et al.*, 2013). However, none of the existing literature considers the retirement problem with two free boundaries in an incomplete market as we do. The approach of Dybvig and Liu (2010) who solved the retirement problem with borrowing constraints in a complete market does not seem to apply to our problem in an incomplete market. Importantly, we successfully show analytical solutions to a retirement problem with borrowing constraints in an incomplete market. Although this paper shares the same fundamental idea with that of Jang *et al.* (2013), we provide a fairly quantitative welfare analysis and implications for private unemployment insurance, which Jang *et al.* (2013) left as an open problem.

We believe that the proposed model will be useful as a tool to study policy implications in pension, insurance, and retirement. By calibrating the wealth-to-income ratios in our model to meet the ratios between family net worth and before-tax family income of the Survey of Consumer Finance for the period 1995-2010, we get two interesting results concerning private unemployment insurance. First, both the RPP of the private unemployment insurance and the IWB from the market innovation might be significantly high for poor people and for people with a low post-retirement leisure preference. Second, under bad market conditions, the RPP and the IWB might increase.

The paper is organized as follows. In Section 2, we describe a financial market with unemployment risks, and specify a retirement problem in an incomplete market and introduce private unemployment insurance. In Section 3, we provide analytical results and investigate the interactions among consumption, risky investment, and voluntary retirement. In Section 4, we derive numerical implications with carefully chosen parameters for optimal strategies and private unemployment insurance. In Section 5, we conclude the paper.

# 2 The Basic Model

## 2.1 Financial Market and Unemployment Risks

Following the conventional models (Merton, 1969, 1971), we assume that there are two assets in the financial market: a bond (or a risk-free asset) and a stock (or a risky asset). The bond price $B(t)$ follows

$$dB(t) = rB(t)dt,$$

where $r > 0$ is a risk-free interest rate. The stock price $S(t)$ is given by the following geometric Brownian motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where $\mu > r$ is the expected rate of the stock return, $\sigma > 0$ is the volatility of the return on the stock, and $W(t)$ is a standard Brownian motion defined on a suitable probability space. The investment opportunity provided by the stock is summarized by the expected stock return $\mu$ and the stock volatility $\sigma$, and assumed to be constant, i.e., $r$, $\mu$, $\sigma$ are positive constants.[5]

We assume that an individual either works full time with income $I_1$ per unit time or retires permanently with income $I_2$ ($I_1 > I_2$). Income after retirement can be annuitized payout from a Social Security program or subsistence such as public welfare or unemployment allowances provided by the government.

The individual is exposed to an unexpected, exogenous, and permanent reduction in future income from $I_1$ to $I_2$ when forced unemployment occurs. To focus on the effect of permanent unemployment shocks on the individual's optimal strategies, the involuntary unemployment event is considered to be permanent; as a result, the risk of unemployment is equivalent to

[5] The assumption of a geometric Brownian motion for the stock price, combined with that the investment opportunity is constant, is standard in the literature on investment (Farhi and Panageas, 2007; Dybvig and Liu, 2010; Jang *et al.*, 2013; Bensoussan *et al.*, 2013). In this paper, we abstract away from other complex issues stemming from a stochastic investment opportunity. For treatment of investigation for the effect of a stochastic investment opportunity, see Chacko and Viceira (2005), and Liu (2007).

the risk of *forced* or *involuntary* retirement. Accordingly, an involuntarily unemployed person can be regarded as an involuntary retiree who is forced to retire when an unemployment shock arrives. Common reasons for involuntary retirement have been poor health condition (43.6%), lay off/dismissal (21.2%), and closed business (10.9%) (Lachance and Seligman, 2008).

Most importantly, the unemployment risk considered in this paper causes a permanent and disastrous labor income shock, which has important influences on an individual's optimal policies over the life-cycle (Viceira, 2001). To treat a disastrous labor income shock, see Cocco *et al.* (2005) who allow for the very small probability of a zero labor income draw of an individual's labor income process. We assume that there are no financial tools (e.g., securities, financial contracts, and insurance contracts) to fully hedge against the unemployment risks. Accordingly, the financial market is essentially *incomplete*. To model the market incompleteness caused by the risk of forced unemployment, we use a Poisson jump process; i.e., the time to a forced unemployment event is distributed according to an exponential distribution. Specifically, the individual can lose her job when an exogenous unemployment shock modeled by the Poisson jump process arrives before a *voluntary* retirement time. More specifically, for time $t \geq 0$

$$\text{Probability of } \{\tau_U \leq t\} = 1 - e^{-\delta t},$$

where $\tau_U$ is the time at which forced unemployment occurs and $\delta > 0$ is an intensity for the unemployment time.[6]

Voluntary retirement and involuntary retirement (or forced unemployment) differently affect an individual's optimal strategies. In the absence of unemployment risks, an individual works full time with labor income $I_1$, which is certain and insurable over the life-cycle. Then she has the following present value of future labor income discounted at the risk-free interest

[6]The unemployment event is assumed to occur at the first jump time $\tau_U$ of a Poisson process with intensity $\delta$, which is independent of the Brownian motion $W(t)$. We can relax the assumption by considering a stochastically-changing $\delta$. We confirm that our main results are robust such change of the assumption. Concerning the modeling of the unemployment event, it is a well-known fact that mortality, disability, retirement, unemployment, and many other events occur at an uncertain time, so the $\tau_U$ following an exponential distribution can aptly capture such uncertain lifetime (Merton, 1971; Richard, 1975; Blanchard, 1985; Viceira, 2001).

rate $r$ (Friedman, 1957; Hall, 1978):[7]

$$E\Big[\int_0^\infty e^{-rt} I_1 dt\Big] = \frac{I_1}{r}, \tag{1}$$

which represents the individual's human wealth. However, in the presence of unemployment risks, the individual encounters an unexpected, exogenous, and permanent reduction in labor income from $I_1$ to $I_2$ when the unemployment event occurs, so she works with a stochastic labor income stream $I(t)$, which evolves by

$$I(t) = \begin{cases} I_1, & \text{if } \ 0 \le t < \tau \wedge \tau_U, \\ I_2, & \text{if } \ t \ge \tau \wedge \tau_U, \end{cases} \tag{2}$$

where $\tau$ is the voluntary retirement time. Then the individual has the following human wealth:

$$E\Big[\int_0^{\tau_U} e^{-rt} I(t) dt\Big] = \frac{1}{r+\delta}\Big(I_1 + I_2\frac{\delta}{r}\Big), \tag{3}$$

which is adjusted by the intensity $\delta$ for unemployment event.

For the limiting case of $\delta = 0$, the individual is not exposed to unemployment risks, as a result, she continuously obtains a constant labor income $I_1$. In this case, the human wealth formulated by (3) reduces to the one given by (1). When she has a possibility of involuntary unemployment, i.e., $\delta > 0$, the amount of income decreases from $I_1$ to $I_2$, so that the human wealth is adjusted by the unemployment intensity $\delta$ with after-retirement income $I_2$. More specifically, as we compared to the human wealth (1), the human wealth (3) is smaller and given by the present value of the sum $(I_1 + I_2\frac{\delta}{r})$ of labor income $I_1$ and after-retirement income $I_2\frac{\delta}{r}$, adjusted by the unemployment intensity $\delta$, discounted at the sum of risk-free interest rate $r$ and intensity $\delta$. For the other limiting case of $\delta = +\infty$, the individual is unemployed and hence she receives after-retirement income $I_2$. In that case, the human wealth is given by $I_2/r$, which is the same as the one of (1) except for that in (1) $I_1$ is replaced by $I_2$.

## 2.2 The Retirement Problem

The retirement problem considered in this paper can be regarded as an extension of the problem explored by Farhi and Panageas (2007), but it allows for an unexpected, exogenous,

[7] The discount rate can be adjusted by considering perceived unemployment risk. For example, we can replace the risk-free interest rate $r$ by the sum of $r$ and the intensity $\delta$ for unemployment time. This change of the discount rate does not alter our main results.

and permanent reduction in future income caused by forced unemployment (or involuntary retirement) event. Importantly, a permanent and drastic decrease of income significantly affects an individual's asset composition (Cocco *et al.*, 2005; Polkovnichenko, 2007; Lynch and Tan, 2011) and voluntary retirement behaviors (Jang *et al.*, 2013; Bensoussan *et al.*, 2013).

An individual has the following logarithmic and time-additive utility function of Cobb-Douglas type (Bensoussan *et al.*, 2013):

$$U(l(t), c(t)) \equiv \frac{1}{a} \ln(l(t)^{1-a} c(t)^a),$$

where $c(t)$ is per-period consumption, $l(t)$ is leisure preference at time $t$, and $0 < a < 1$ is the weight for consumption. We assume that the individual either works full time with income $I_1$ per unit time or retires permanently, and enjoys leisure $l(t) = \underline{l}$ while working and $l(t) = \bar{l}$ ($\bar{l} \geq \underline{l} > 0$) when she retires.[8] We also assume that the individual receives a post-retirement income $I_2$ ($I_1 > I_2$) per unit time. If we normalize pre-retirement leisure $\underline{l}$ as 1, the utility function during working status is given by

$$U(1, c(t)) = \ln c(t).$$

The utility function after (voluntary or involuntary) retirement follows

$$U(\bar{l}, c(t)) = \ln\{\bar{l}^{1/a-1} c(t)\}.$$

For the notational simplicity, we introduce the following constant $K$:

$$K \equiv \bar{l}^{1/a-1} > 1.$$

Then the constant $K$ represents post-retirement leisure preference; i.e., an individual enjoys more leisure as $K$ increases. The constant $K$ also reflects that fact that the marginal utility of consumption is larger after retirement than before retirement. This preference for leisure after

[8]Economists usually assume that the income rate of $I_1$ is equivalent to $\omega(\bar{l} - \underline{l})$ during working, if the wage rate $\omega > 0$ is constant. In this paper, labor supply can be adjusted only along the extensive margin. If individuals can adjust hours of work on the job (Bodie *et al.*, 1992), some results might be modified. To obtain a more tractable life-cycle model, we follow Farhi and Panageas (2007) and Dybvig and Liu (2010) who assume that the individual's retirement decision is controlled by her rather than labor flexibility along the intensive margin. We leave the retirement problem of an individual who can respond to unemployment risks by changing her labor hours as an open problem.

retirement results from a disutility of work (Choi and Shim, 2006), or household production, or cost savings (Dybvig and Liu, 2010). For instance, retirement may allow sufficient time to enjoy leisure (such as shopping for bargains, preparing meals, and taking a cruise etc); i.e., time spent away from business work, domestic chores, and education.

The individual's wealth process $X(t)$ with initial wealth $X(0) = x$ should satisfy

$$dX(t) = \big(rX(t) - c(t) + I(t)\big)dt + \pi(t)\sigma(dW(t) + \theta dt), \quad \text{for} \quad t \geq 0, \tag{4}$$

where $\pi(t)$ is the dollar amount invested in the stock, $\theta$ denotes the *Sharpe ratio*, $(\mu - r)/\sigma$, and $I(t)$ is a stochastic labor income stream formulated by (2). The individual accumulates wealth at the rates of $(rX(t) - c(t) + I(t))$. She consumes at the rate equal to $c(t)$ and obtains risk-free interests in proportional to wealth by the bond investment. Most importantly, the individual's income stream is stochastic due to an unexpected, exogenous, and permanent reduction in future income induced by forced unemployment event, accordingly the wealth accumulation also varies according to the changes of income level $I(t)$.

The individual chooses to allocate her wealth between a risk-free bond and a risky stock. When the individual invests in the stock market, she is exposed to the market risk from her stock holdings, i.e., bears stochastic fluctuations of wealth caused by the term that involves the Brownian motion $W(t)$. Specifically, the wealth randomly changes at the rate $\pi(t)\sigma$, which is the product of the dollar amount $\pi$ invested in the stock and the stock volatility $\sigma$ that denotes the standard deviation of the return on the stock. Risk taking is compensated for by a positive risk premium, so that the rate of wealth accumulation is increased by $\pi(t)\sigma\theta = \pi(t)(\mu - r)$, the product of the amount $\pi$ invested in the stock and the risk premium $(\mu - r)$, as we compared to the case where the individual invests only in the risk-free bond.

Throughout the paper, we impose borrowing constraints prior to voluntary or involuntary retirement as the following:

$$X(t) \geq 0 \quad \text{for} \quad 0 \leq t < \tau \wedge \tau_U. \tag{5}$$

The wealth constraint has been considered in a complete market in the absence of labor income risk to investigate its impact on an individual's optimal strategies (Farhi and Panageas, 2007; Dybvig and Liu, 2010; Jang *et al.*, 2013). The crucial point here is that the nonnegative wealth constraint would be significant in an incomplete market, especially when labor income risk exists. Most papers regarding life-cycle consumption and portfolio choice emphasize the nonnegative wealth constraint; i.e., that an individual cannot borrow money with her unsecured or uncollateralized future income at all times. For treatments that consider both labor income risk and borrowing constraints, see Viceira (2001), Cocco *et al.* (2005), Polkovnichenko (2007), Benzoni *et al.* (2007), Wachter and Yogo (2010), and Munk and Sørensen (2010).

In this paper, an individual is assumed to have after-retirement income $I_2$, which can be annuitized payout from a Social Security program or subsistence such as public welfare or unemployment allowances provided by the government. Then the individual has the following present value of after-retirement income discounted at the risk-free interest rate:

$$E\Big[\int_0^{\infty} e^{-rt} I_2 dt\Big] = \frac{I_2}{r}.$$

Thus, we impose a natural wealth constraint after voluntary or involuntary retirement as the following:

$$X(t) \geq -\frac{I_2}{r} \quad \text{for} \quad t \geq \tau \wedge \tau_U.$$

That is, we allow for borrowing with secured or collateralized after-retirement income.

The retirement problem is to maximize the individual's life-time utility function of consumption by controlling per-period consumption $c$, risky investment $\pi$ and voluntary retirement time $\tau$ in the presence of risk of forced unemployment, i.e., to find the following individual's value function:

$$\Phi(x) \equiv \max_{(c,\pi,\tau)} E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}\ln c(t)dt + e^{-\beta(\tau\wedge\tau_U)}\int_{\tau\wedge\tau_U}^{\infty} e^{-\beta(t-\tau\wedge\tau_U)}\ln\left(Kc(t)\right)dt\Big], \tag{6}$$

where $x \geq 0$ is the initial wealth of the individual, $E$ is the expectation taken at time 0 and $K > 1$ is the preference for leisure.[9] The random variable $\tau_U$ represents the time at which forced unemployment time occurs, and $\beta > 0$ is the individual's subjective discount rate. In (6) we consider a risk-averse individual and assume that she has a log-type utility

[9] We consider an infinite-horizon life-cycle model; as a result, we overestimate the effects of unemployment risks in that our representation of forced unemployment is more painful than it actually is because unemployed people undergo a drastic decrease of income forever. Furthermore, to increase the realism of the life-cycle model, a finite-horizon model with time-varying unemployment probability should be considered, because unemployment risks fluctuate significantly at business cycle frequencies. However, such a modification makes the retirement problem difficult to solve and is far beyond the scope of the current paper.

for her consumption. The second part of the right hand side in (6) is equivalent to the value function of an individual in Merton's (1969) problem with income stream $I_2$ over an infinite investment horizon. For simplicity, we assume that the individual has no bequest motive. The presence of the bequest motive reinforces the effect of an unexpected, exogenous, and permanent reduction in future income when forced unemployment occurs.

### 2.3 Private Unemployment Insurance

The retirement problem considered in the previous sections is an optimal consumption and portfolio selection problem for an individual who wants to retire someday in the future, but who is exposed to risk of forced unemployment and is subject to borrowing constraints. Because we assume that unemployment risks cannot be diversified away, the financial market considered until now has been incomplete.

Specifically, borrowing the idea of Jang *et al.* (2013),[10] we construct a complete market by considering private unemployment insurance with a premium rate of $\delta\big(\overline{X}(t)-X(t)\big)$ in the financial market. Then the wealth process $X(t)$ of the individual under its coverage should satisfy

$$dX(t) = \begin{cases} \big\{rX(t) - c(t) + I_1 - \delta\big(\overline{X}(t) - X(t)\big)\big\}dt + X(t)\pi(t)\sigma(dW(t) + \theta dt), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \;\; 0 \le t < \tau \wedge \tau_U, \\ \big(rX(t) - c(t) + I_2\big)dt + X(t)\pi(t)\sigma(dW(t) + \theta dt), \quad \text{for} \;\; t \ge \tau \wedge \tau_U, \end{cases} \tag{7}$$

where $\overline{X}(t)$ is the *potential retirement wealth process*; the individual pays $\delta\big(\overline{X}(t)-X(t)\big)$ per unit time and obtains the lump-sum unemployment coverage of $(\overline{X}(t) - X(t))$ at the forced unemployment time. If a forced unemployment event occurs before the voluntary retirement time, the individual receives $(\overline{X}(t)-X(t))$, therefore, her wealth level immediately jumps from $X(t)$ to $\overline{X}(t)$, then she can arrive at the voluntary retirement wealth level which was planned initially by utilizing the traditional optimal consumption and risky investment strategies (Merton, 1969), that is, $\overline{X}(t) = X(t)$ for $t \ge \tau$ in this case.

The retirement problem for the individual who has insurance coverage is now defined as

$$\Psi(x) \equiv \max_{(c,\pi,\overline{X},\tau)} E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}\ln c(t)dt + e^{-\beta(\tau\wedge\tau_U)}\int_{\tau\wedge\tau_U}^{\infty} e^{-\beta(t-\tau\wedge\tau_U)}\ln\left(Kc(t)\right)dt\Big], \tag{8}$$

[10] Jang *et al.* (2013) introduced fully personalized unemployment insurance to hedge against unemployment risks. Our paper considers a private unemployment insurance, the same as theirs.

and an analytical solution can be obtained by utilizing the dynamic programming approach.[11]

# 3 Analytical Results

In this section, we provide analytical results for an individual's optimal retirement behavior, and optimal consumption and risky investment strategies in the presence of risk of forced unemployment.

## 3.1 Optimal Retirement Behavior

To begin, we use the following lemma to reformulate the value function formulated by (6).

**Lemma 3.1** *The value function (6) can be rewritten by*

$$\Phi(x) = \max_{(c,\pi,\tau)} E\Big[\int_0^\tau e^{-(\beta+\delta)t}\Big\{\ln c(t) + \delta U_2(X(t))\Big\}dt + e^{-(\beta+\delta)\tau}U_2(X(\tau))\Big], \tag{9}$$

*where*

$$U_2(z) = \frac{1}{\beta}\Big[\ln\Big\{\beta\Big(z + \frac{I_2}{r}\Big)\Big\} + \frac{1}{\beta}\Big(r + \frac{\theta^2}{2} - \beta(1 - \ln K)\Big)\Big].$$

**Proof.** See Appendix. **Q.E.D.**

The term $U_2(\cdot)$ in Lemma 3.1 represents the value function of an individual who obtains income at the rate of $I_2$ infinitely. Note that the value function reformulated by (9) shows that in the presence of risk of forced unemployment, an individual optimally considers both her consumption and her wealth at the time of forced unemployment. More specifically, the term $\delta U_2(X(t))$ in the first integral term in the right hand side of (9) captures the utility value of wealth after involuntary retirement. The term is the product of the intensity $\delta$ for the unemployment time and the maximized value of the individual's utility after the unemployment event.

For the extreme case of $\delta = 0$, i.e., the individual is not exposed to risk of forced unemployment, she maximizes a utility function of intermediate consumption and a utility function of wealth at voluntary retirement time (Farhi and Panageas, 2007). For the other limiting case of $\delta = +\infty$, i.e., the individual is unemployed, the value function $\Phi(x)$ given by (9) reduces to the value function $U_2(x)$ obtained under the assumption that the individual's income is given by the rate equal to $I_2$ infinitely (Merton, 1969).

[11] See Appendix 6.6 for the details.

A retirement problem with borrowing constraints generally corresponds to a variational inequality with two free boundaries (Farhi and Panageas, 2007; Dybvig and Liu, 2010; Jang *et al.*, 2013). However, this situation only holds in a complete market. Most importantly, our paper is distinguished from the existing literature in that we consider an incomplete market in which unhedgeable unemployment risk and borrowing constraints are concurrent. Now we provide a lemma that clarifies the relationship between the optimal retirement problem in an incomplete market and the optimal stopping problem.

**Lemma 3.2** *The value function reformulated by (9) satisfies the optimal stopping problem given by the following variational inequality: for any* $x \geq 0$*,*

$$\begin{cases} (\beta+\delta)\phi(x) - (rx + I_1)\phi'(x) + \dfrac{\theta^2}{2}\dfrac{\phi'(x)^2}{\phi''(x)} + 1 + \ln\phi'(x) \geq \delta U_2(x), \\ \phi(x) \geq U_2(x), \\ \left[(\beta+\delta)\phi(x) - (rx + I_1)\phi'(x) + \dfrac{\theta^2}{2}\dfrac{\phi'(x)^2}{\phi''(x)} + 1 + \ln\phi'(x) - \delta U_2(x)\right]\Big(\phi(x) - U_2(x)\Big) = 0. \end{cases} \tag{10}$$

**Proof.** See Appendix. **Q.E.D.**

The optimal retirement strategy for an individual consists of two regions: a continuation region in which the individual's optimal choice is to work; and a stopping region in which she should retire voluntarily. The first inequality in the variational inequality (10) shows that the equality holds in the continuation region and the strict inequality holds in the stopping region. Specifically, the equality is the Hamilton-Jacobi-Bellman equation that was derived when optimality conditions to consumption and risky investment were applied to an optimal consumption and portfolio choice problem (Merton, 1969). The strict inequality in the second inequality denotes the case where the individual's value function with a retirement option is larger than the value function after voluntary or involuntary retirement. In this case, the individual is in the continuation region, so optimally she continues to work. If the individual's value function before voluntary retirement approaches the value function after retirement, then the individual is in the stopping region, and should retire voluntarily. The third equality in (10) is necessary because the first and second inequalities cannot hold simultaneously, i.e., the stopping region and the continuation region characterized by the first strict inequality and the second strict inequality, respectively, cannot coexist.

The continuation and stopping regions are determined by the so-called *critical wealth level*; if the individual has more than this wealth, her optimal decision is to retire. We conjecture that our optimal stopping problem formulated by the variational inequality (10) can be solved by finding two free boundaries; one is an optimal stopping boundary $\hat{x}$, i.e., the critical wealth level; the other is a free boundary that corresponds to the nonnegative wealth constraint (5). These two boundaries are determined by value-matching and smooth-pasting conditions. Specifically, we consider the following problem:

$$\begin{cases} (\beta+\delta)\phi(x) - (rx+I_1)\phi'(x) + \dfrac{\theta^2}{2}\dfrac{\phi'(x)^2}{\phi''(x)} + 1 + \ln\phi'(x) = \delta U_2(x), \quad 0 \le x < \hat{x}, \\ \phi(x) = U_2(x), \quad x \ge \hat{x}, \\ \phi(\hat{x}) = U_2(\hat{x}), \\ \phi'(\hat{x}) = \dfrac{1}{\beta}\dfrac{1}{\hat{x}+I_2/r}. \end{cases} \tag{11}$$

In fact, the problem stated above is exactly same as the one given by Bensoussan *et al.* (2013) except for that initial wealth is always larger than or equal to zero, which represents the nonnegative wealth constraint.[12]

### 3.2 Optimal Consumption and Risky Investment Strategies

To derive the optimal consumption and risky investment strategies of an individual who is faced with a risk of forced unemployment, we must find a solution to the free boundary problem (11). Solving the highly nonlinear differential equation given by the first equation in (11) analytically seems to be hardly possible. In general optimal consumption and portfolio choice problems in an incomplete financial market are difficult to solve. In this paper the market incompleteness arises from a down-jump of income at the time of forced unemployment; to overcome this problem, Jang *et al.* (2013) introduced a personalized private unemployment insurance and constructed a complete market, so they successfully solved the optimal retirement problem by using the martingale approach of Karatzas and Shreve (1998).

Distinct from Jang *et al.* (2013), we use the conventional dynamic programming approach to solve the optimal retirement problem in an incomplete market. Actually, the approach shares the same fundamental idea with Bensoussan *et al.* (2013) who solved the optimal

[12]If we find $\phi(x)$ such that it is $C^1$ and piecewise $C^2$, satisfying the inequalities given in the variational inequality (10). Also, the solution to (10) is equivalent to the value function $\Phi(x)$ given by (9). For the details, see Appendix.

retirement problem with unemployment risks. However, this paper differs significantly frmo theirs in the way in which solutions were derived. Importantly, we are required to determine two free boundaries rather than one free boundary. The two free boundaries are determined by the appropriate value-matching and smooth-pasting conditions.

We modify the idea of Bensoussan *et al.* (2013) and apply it to our optimal retirement problem that includes both unhedgeable risk and borrowing constraints. We provide the following lemma that restates the free boundary problem (11) by using the modified approach.

**Lemma 3.3** *The Hamilton-Jacobi-Bellman equation formulated by the first relationship in (11) is modified as the following nonlinear equation:*

$$-\frac{1}{2}\theta^2\lambda^2 G''(\lambda) - \lambda G'(\lambda)(\theta^2+\beta+\delta-r) + rG(\lambda) + \frac{\delta}{\beta}\frac{G'(\lambda)}{G(\lambda)-I_1/r+I_2/r} = \frac{1}{\lambda}, \quad \underline{\lambda} < \lambda < \bar{\lambda}, \tag{12}$$

*where $G$ is a convex-dual function of the value function $\phi$, $\lambda$ is a marginal value of the value function $\phi$, and $\bar{\lambda}$ and $\underline{\lambda}$ are two free boundaries to be determined by the value-matching and smooth-pasting conditions, corresponding to the borrowing constraints and critical wealth level, respectively.*

**Proof.** See Appendix. **Q.E.D.**

Function $G$ is called the convex-dual function. We verify that the $G$ is monotonically-decreasing with respect to an increase in initial wealth $x$. Further, the function $G$ has an implicit relationship with the marginal value of the value function $\phi$ as the following:

$$G\big(\lambda(x)\big) \equiv G\big(\phi'(x)\big) = x + I/r.$$

Therefore, $G$ is the dual function of the value function $\phi$ such that it is increasing and concave in initial wealth. Under reasonable parameter values, the dual function $G$ can be verified to be convex in initial wealth.

In Lemma 3.3, the free boundary $\underline{\lambda}$ has an inverse relationship with the critical wealth level $\hat{x}$ as follows: $\underline{\lambda} = \frac{1}{\beta(\hat{x}+I_2/r)}$. To determine the free boundary $\bar{\lambda}$ we use the value-matching and smooth-pasting conditions. More specifically, we use the following boundary conditions of $\phi$ and $\phi'$ at $\hat{x}$:

$$\phi(\hat{x}) = U_2(\hat{x}), \quad \phi'(\hat{x}) = \frac{1}{\beta}\frac{1}{\hat{x}+I_2/r}.$$

As stated in Lemma 3.3, the another free boundary $\bar{\lambda}$ corresponds to the borrowing constraints and should be determined by the appropriate value-matching and smooth-pasting conditions.

The first natural condition can be imposed as the following:

$$G(\bar{\lambda}) \equiv G\big(\phi'(0)\big) = I_1/r,$$

by utilizing the implicit relationship between the convex-dual function $G$ and the marginal value $\phi'$ of the value function $\phi$. Further, we impose one more constraint

$$G'(\bar{\lambda}) = 0,$$

which gives the zero risky investment as the individual's initial wealth approaches zero.[13]

For the next, we present an important theorem that suggests an analytic solution to the Hamilton-Jacobi-Bellman equation (12).

**Theorem 3.1** *An analytic solution to the Hamilton-Jacobi-Bellman equation (12) is given by*

$$\begin{aligned} G(\lambda) = \quad & \frac{1}{\lambda(\beta+\delta)} + B(\underline{\lambda})\lambda^{-\alpha_\delta} + B^*(\bar{\lambda})\lambda^{-\alpha_\delta^*} \\ & + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\Big[(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_{\underline{\lambda}}^{\lambda} \mu^{\alpha_\delta - 2}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu \\ & + (\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\bar{\lambda}} \mu^{\alpha_\delta^* - 2}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu\Big], \end{aligned} \tag{13}$$

*where $\alpha_\delta > 0$ and $\alpha_\delta^* < 0$ are the two solutions to the following characteristic equation:*[14]

$$F(\alpha;\delta) \equiv -\frac{1}{2}\theta^2\alpha(\alpha-1) + \alpha(\beta+\delta-r) + r = 0,$$

*and $B(\underline{\lambda})$ and $B^*(\bar{\lambda})$ are two constants to be determined according to the value-matching and smooth-pasting conditions.*

**Proof.** See Appendix. **Q.E.D.**

The optimal consumption and portfolio choice of an individual before voluntary retirement in an incomplete financial market can be stated as a functional of the convex-dual function $G$.

---

[13]If initial wealth approaches zero, then the individual cannot consume and invest in the stock market, i.e., in this case consumption $c$ and risky investment $\pi$ should be zero. Note that optimality conditions for consumption and risky investment are given by

$$c^*(t) = \frac{1}{\phi'(x)} = \frac{1}{\lambda^*(x)}, \quad \text{and} \quad \pi^*(t) = -\frac{\theta}{\sigma}\frac{\phi'(x)}{\phi''(x)} = -\frac{\theta}{\sigma}\lambda^*(x)G'\big(\lambda^*(x)\big),$$

accordingly the constraint of $G'(\bar{\lambda}) = 0$ implies the zero risky investment at zero wealth level.

[14]We impose parameter conditions such that $(\beta+\delta-r+\frac{1}{2}\theta^2)^2 + 2\theta^2 r > 0$ in order to ensure that there are two distinct solutions to the characteristic equation.

**Theorem 3.2** *The optimal consumption $c^*$ and risky investment $\pi^*$ before voluntary retirement in an incomplete financial market are given by*

$$
\begin{aligned}
c^*(t) \;&= (\beta+\delta)\Big(x+\frac{I_1}{r}\Big) - (\beta+\delta)B(\underline{\lambda})\lambda^*(x)^{-\alpha_\delta} - (\beta+\delta)B^*(\bar{\lambda})\lambda^*(x)^{-\alpha_\delta^*} \\
&\quad - \frac{2\delta(\beta+\delta)}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_{\underline{\lambda}}^{\lambda^*(x)}\mu^{\alpha_\delta-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu \\
&\quad + (\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\bar{\lambda}}\mu^{\alpha_\delta^*-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu\Big],
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\pi^*(t) \;&= \frac{\theta}{\sigma}\frac{1}{\lambda^*(x)(\beta+\delta)} + \frac{\theta}{\sigma}\alpha_\delta B(\underline{\lambda})\lambda^*(x)^{-\alpha_\delta} + \frac{\theta}{\sigma}\alpha_\delta^* B^*(\bar{\lambda})\lambda^*(x)^{-\alpha_\delta^*} - \frac{2\delta}{\sigma\theta\beta\lambda^*(x)}\ln\Big\{\beta\Big(x+\frac{I_2}{r}\Big)\Big\} \\
&\quad + \frac{2\delta}{\sigma\theta(\alpha_\delta-\alpha_\delta^*)\beta}\Big[\alpha_\delta(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_{\underline{\lambda}}^{\lambda^*(x)}\mu^{\alpha_\delta-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu \\
&\quad + \alpha_\delta^*(\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\bar{\lambda}}\mu^{\alpha_\delta^*-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu\Big],
\end{aligned} \tag{15}
$$

*where $\lambda^*(x)$ is a decreasing function of wealth $x$.*

**Proof.** See Appendix 6.7. **Q.E.D.**

When no forced unemployment event occurs (i.e., $\delta = 0$), our consumption strategy becomes similar to the form given by Farhi and Panageas (2007) and Dybving and Liu (2010):

$$c(t) = \beta\Big(x+\frac{I_1}{r}\Big) - \beta B(\underline{\lambda})\lambda^*(x)^{-\alpha_\delta} - \beta B^*(\bar{\lambda})\lambda^*(x)^{-\alpha_\delta^*}. \tag{16}$$

Under the assumption that

$$\ln K > \underline{\lambda}(I_1-I_2)\Big(1+\frac{\theta^2\alpha_0^*}{2r}\Big),$$

$B(\underline{\lambda}) > 0$, and thus, the second term of the right hand side of (16), which is closely associated to voluntary retirement, has a negative value. This implies that an individual who is considering voluntary retirement consumes relatively less than in the classical portfolio selection problems (Merton, 1969), because the desire for voluntary retirement results in a cutback in consumption to accumulate wealth. The effect of reducing consumption seems to increase as her wealth increases, or equivalently as $\lambda^*$ decreases; this conclusion is also consistent with the results in Farhi and Panageas (2007) and Dybvig and Liu (2010) in which individuals are willing to significantly decrease their consumption as voluntary retirement time approaches.

The third term of the right hand side of equation (16) reveals the effect of the borrowing constraints on consumption. Under the assumption that $B^*(\bar{\lambda}) > 0$, or put differently,

$$\frac{\alpha_0 I_1}{r}\bar{\lambda} > \frac{\alpha_0 - 1}{\beta},$$

the individual consumes less in the presence of the borrowing constraints than in their absence. The effect of the constraint on the individual's consumption seems to become more significant as her wealth decreases, and this trend implies that poor people, who mostly have a high possibility of binding the wealth constraint, tend to reduce consumption more than do rich people.

The effect of risk of forced unemployment on optimal consumption is revealed by the last two terms in the bracket of the right hand side of equation (14). As the individual's wealth approaches the critical wealth level $\hat{x}$ (or $\lambda^*$ approaches $\underline{\lambda}$), the first term in the bracket vanishes and the second term becomes a positive fixed value. Therefore, near retirement, individuals who are exposed to risk of unemployment might consume more than do individuals who are not exposed to this risk. In contrast, for a sufficiently small wealth level, the first term remains negative but the second one disappears. As a result, unemployment risks reinforce the negative effect of borrowing constraints on the individual's consumption, and accordingly, induce poor people who have the risks to consume far less than do poor people who do not have unemployment risks.

Concerning optimal risky investment $\pi^*$, for $\delta = 0$ we get

$$\pi(t) = \frac{\theta}{\beta\sigma}\frac{1}{\lambda^*(x)} + \frac{\theta}{\sigma}\alpha_0 B(\underline{\lambda})\lambda^*(x)^{-\alpha_0} + \frac{\theta}{\sigma}\alpha_\delta^* B^*(\bar{\lambda})\lambda^*(x)^{-\alpha_0^*}. \tag{17}$$

The second term of the right hand side of (17) represents the positive effect of voluntary retirement on the individual's risky investment and becomes increasingly significant as her wealth approaches the critical wealth level $\hat{x}$. This trend implies that a strong desire for voluntary retirement makes the individual increase her investment in the risky asset; this conclusion is comparable to the findings of Farhi and Panageas (2007) and Dybvig and Liu (2010). In contrast, the negative effect of the wealth constraints on risky investment becomes increasingly significant as wealth decreases. Hence, poor people invest less in the risky asset than do rich people.

The effect of unemployment risks on optimal risky investment is mainly represented in the fourth term and the last two terms in the bracket of the right hand side of equation

(15). The fourth term in (15), which has a negative value, allows the individual to reduce her risky investment, whereas the last two terms in the bracket have positive values, which make her increase her investment in the risky asset. Combining all the effects, the adjustment of risky portfolio can be either larger or smaller in the presence of unemployment risks than in their absence. When the individual has sufficient wealth, her optimal portfolio choice in the presence of unemployment risks is not much different from that in their absence; i.e., a sufficiently rich individual is little affected by the presence of unemployment risks when formulating risky investment strategy.

# 4 Numerical Implications

We analyze the sensitivity of the individual's optimal consumption and risky investment strategies, and optimal retirement behaviors with respect to the changes of parameter values.[15]

## 4.1 Baseline Parameters

The baseline market parameters are set as follows: $r = 3.71\%$, the annual rate from rolling over of 1-month T-bills during the time period of $1926 - 2009$;[16] $\mu = 11.23\%$ and $\sigma = 19.54\%$, the return and standard deviation, respectively, of a portfolio consisting of the world's large stocks during the same time period;[17] $I_1 = 1$, the annual rate of income prior to retirement; $K = 3$ by adopting the parameter value used in Dybvig and Liu (2010), the post-retirement leisure preference; and $I_2 = 0.10$ by following Lynch and Tan (2011),[18] the post-retirement income that can be annuitized payout from a Social Security program or subsistence such as public welfare or unemployment allowances provided by the government.

We assume that the subjective discount rate $\beta$ equals to the risk-free interest rate $r$. This assumption can be rational in the sense that in determining the appropriate discount rate, we have considered perceived unemployment risks. More specifically, we replaced $\beta$ by $\beta + \delta$, which is the subjective discount rate adjusted by the intensity $\delta$ for the unemployment time.

[15]We exploit the iterative procedure in Appendix 6.7 to present graphical illustration and more detailed discussion of our main results given in Theorem 3.2.

[16]Source: Bureau of Labor Statistics.

[17]See pp. 170 of Bodie, Kane, and Marcus (2011)

[18]They believed that unemployment state pays 10% of permanent labor income and utilize it to obtain economic results.

This is reflected by the discount rate used in the value function (9).

In this paper, the retirement state occurs in two ways: First, an individual optimally enters retirement when she reaches a certain wealth threshold (the so-called critical wealth level); or she is forced to retire because of unemployment shock. In either case, the retirement status is *irreversible*.[19] Most importantly, we allow for the small possibility of a disastrous labor income shock that is modeled as a down-jump of income from $I_1$ to $I_2$ with some probability. Carroll (1997) used this type disastrous labor income shock and Cocco *et al.* (2005) modeled the disastrous shock as the presence of a 0.5% probability of zero income at each period in a life cycle (discrete-time and finite horizon) framework. Therefore, we also set $\delta$ to 0.5%. This simply means that the individual can become employed and her income is modeled as being $I_1$ with an annual probability of 99.5%. Otherwise, she becomes unemployed permanently and her income is modeled as following $I_2$ with an annual probability of 0.5%.[20]

### 4.2 Optimal Consumption and Portfolio Selection

Empirical evidence available suggests that an individual's labor income is exposed to both permanent and transitory shocks (MaCurdy, 1982; Abowd and Card, 1989, Carroll, 1992). Importantly, income fluctuations induced by such income shocks cause an individual to accumulate savings as a precaution. Caballero (1990, 1991), Weil (1993) found a constant precautionary savings demand due to the assumption of CARA utility, but Wang (2006) found stochastic precautionary savings by allowing for conditional heteroskedasticity, i.e., the conditional variance of income changes is an affine function of the labor income.

The amounts of optimal consumption and risky investment decrease as the intensity $\delta$ of forced unemployment increases (Table 1). Importantly, individuals could be significantly affected by risk of forced unemployment even if the possibility of unemployment is very small. For example, the individual with wealth $\hat{x} - 45$ reduces consumption by 16.42% as $\delta$ increases

[19] Viceira (2001) also considered irreversible retirement status that occurs with a constant probability. An interesting extension of this paper is to allow an individual to reenter the workforce at a reduced income after retirement.

[20] The annual unemployment probability is given by

$$\int_0^1 \delta e^{-\delta t} dt = 1 - e^{-\delta}.$$

When we set $\delta = 0.5\%$, we obtain an annual probability of 0.5%.

from 0 to 0.5%. Further, as income shocks become increasingly persistent (i.e., $\delta$ increases), the amount of consumption also decreases; this trend demonstrates that the incentive to engage in precautionary saving increases. This is an empirically testable prediction.

**[Insert Table 1 here.]**

Economists have reached a consensus that increased flexibility in labor supply induces increase in stock holdings (Bodie *et al.*, 1992; Farhi and Panageas, 2007; Dybvig and Liu, 2010), whereas increase in labor income risks tends to decrease risky investment (Koo, 1998; Heaton and Lucas, 2000; Jang *et al.*, 2013). We believe that undiversifiable unemployment risk is one of the main sources of background risk (Kimball, 1993) and show that the individual willingly takes lower risk when investing in financial assets in the face of such unhedgeable risk.

For example, the individual with wealth $\hat{x} - 45$ reduces risky investment by 23.80% as $\delta$ increases from 0 to 0.5%; this trend implies that individuals with a small wealth reduce their risky investment more than do rich people (Table 1). Some important implications are that a large wealth is a good buffer against disastrous labor income shocks, and that risky asset may not be a good substitute for the defaultable labor income; as a result, to reduce her risk exposure the individual reduces her investment in the risky asset.

The amounts of optimal consumption and risky investment are expected to increase as an individual's wealth increases (Figure 1). An interesting feature that has been obtained in previous work is that retirement flexibility reduces consumption and increases investment in a risky portfolio (Farhi and Panageas, 2007; Dybvig and Liu, 2010; Jang *et al.* (2013), and Bensoussan *et al.*, 2013). We show that this conclusion still holds *even when the risk of forced unemployment and the borrowing constraints are considered jointly.* In particular, due to the individual's strong desire for voluntary retirement, the optimal consumption ratio and optimal risky investment ratio decrease and their rates of decrease decline as wealth approaches the critical wealth level $\hat{x}$.

Further, consumption and investment ratios decrease more rapidly as the expected rate of stock return $\mu$ decreases. The intuition is as follows: voluntary retirement becomes increasingly attractive to the individual if her investment opportunity degrades. The opposite results are true for volatility parameter $\sigma$, which leads us to the same economic implications (Figure 2).

**[Insert Figure 1 and 2 here.]**

Consumption decreases and investment in the risky asset increases as the individual's leisure demand increases (Figure 3). Intuitively, she can enter early retirement by reducing consumption and pursuing higher expected portfolio returns. Furthermore, for an individual with a high leisure preference, the optimal strategy may be to reduce her consumption rate more rapidly and her investment ratio less rapidly when imminent voluntary retirement is anticipated than when it is not. This is because a high leisure demand boosts preference for retirement, so the individual is willing to reduce consumption and increase stockholdings.

**[Insert Figure 3 here.]**

### 4.3 Optimal Retirement Strategy

The proportion of workers who opted for early retirement sharply increased between 1995 and 2000 owing to the booming U.S. stock market (Gustman and Steinmeier, 2002). The rationale is that investing for early retirement reinforces an individual's behavior that increases her savings and stock market exposure compared to the case in which early retirement is not allowed (Farhi and Panageas, 2007). As a result, the booming stock market is accompanied by the increased proportion of voluntary retirees. This observation suggests that meaningful gauge of rising confidence in the stock market can help to understand the increase in the number of individuals who chose early retirement.

However, in 2000, "Issues in Labor Statistics" published by the US Bureau of Labor Statistics showed that the number of voluntary retirees demonstrate a counter-cyclical pattern: the number increases during down-markets and decreases during up-markets. Jang *et al.* (2013) confirmed this counter-cyclical pattern in a complete market, and Bensoussan *et al.* (2013) confirmed it in an incomplete market.

Our analysis complements existing understanding of voluntary retirement behaviors in two ways. First, we show that an individual enters retirement when she reaches a certain wealth threshold (i.e., the *critical wealth level*); we will illustrate this trend clearly in the next subsection. Along with the wealth threshold an intuitive prediction can be made. In addition to determining financial resources for future consumption, the individual's accumulated wealth controls her optimal time of retirement. Further, we study how the critical wealth level depends on the level of investment opportunity, preference for leisure after retirement, and unemployment risks. Second, we analyze the value of income with respect to changes in various parameter values by using a useful concept of an *implicit value of income*: i.e., the marginal rate of substitution between an individual's income and financial wealth. Then the implicit value of income is the individual's subjective marginal value of her labor, i.e., a criterion for the individual's optimal retirement decision; a higher value than the implicit value of after-retirement income implies that the individual is willing to work, thereby delaying retirement, but a lower value implies that the individual is willing to enter voluntary retirement.

#### 4.3.1 Critical Wealth Level

For an individual with strong demands for post-retirement leisure, the optimal choice is to enter voluntary retirement as soon as her wealth reaches $\hat{x}$. This is the wealth at which utility gains due to the increase in leisure are equal to utility losses due to the decrease in income after retirement.

The critical wealth level at which an individual chooses to retire increases as expected return rates increase or as stock volatility decreases (or both), because these changes increase the attractiveness of investing in financial assets to increase wealth, rather than retiring early to enjoy leisure (Table 2). Increase in preference for leisure after retirement (i.e., the increase in $K$) may have the opposite effect.

**[Insert Table 2 here.]**

#### 4.3.2 Implicit Value of Income

The post-retirement actuarially-fair value (PRAV) of income should be

$$E\Big[\int_0^\infty I_2 e^{-rt} dt\Big] = \frac{I_2}{r},$$

which will be regarded as a benchmark measure of implicit value of income in this subsection. Following Koo (1998), we define an implicit value of labor income as the marginal rate of substitution between income and financial wealth.

**Definition 4.1** *Let $\Phi(x; I_1, \delta)$ be the value function described in section 2.2 given that the annual rate of income prior to retirement is $I_1$ and the unemployment intensity is $\delta$. Then* *the implicit value of income is defined as*

$$\frac{\partial \Phi(x; I_1, \delta)}{\partial I_1} \Big/ \frac{\partial \Phi(x; I_1, \delta)}{\partial x}.$$

The implicit value of income increases as wealth increases up to a somewhat small level, then starts to decrease (Figure 4). This hump-shaped pattern implies that labor income may be attractive to the poor, but may be gradually less attractive as the individual's wealth increases, as a result, individuals enter voluntary retirement when the desire for early retirement dominates the utility gains from the labor income. Accordingly, the intersection point of the implicit value of income and the PRAV of income is the critical wealth level, and as expected the intersection happens at a smaller wealth level in the presence of risk of forced unemployment than in its absence.

[**Insert Figure 4 here.**]

The relationship between implicit value of income and investment opportunity (Figure 5) reveals that the poor might prefer a high risk premium from the stock investment over uninsurable or unhedgeable labor income, whereas the opposite might be true for the rich who are about to retire. Uninsurable or unhedgeable labor income can not be a good substitute for the risky stock; labor income will be less favorable to the poor than to the rich when the risky asset guarantees a high risk premium, i.e., if the financial market has a good investment opportunity.

[**Insert Figure 5 here.**]

Intuitively, a low post-retirement leisure preference induces a high implicit value of labor income. Therefore, as an individual's post-retirement leisure preference decreases, her willingness to delay retirement to increase utility gains from the labor income increases (Figure 6).

[**Insert Figure 6 here.**]

### 4.4 Certainty Equivalent Wealth Gain for Unemployment Risks

We define a certainty equivalent wealth gain (CEWG) for unemployment risks as the largest wealth that an individual is willing to give up in exchange for making the individual's income unemployment-free. In this sense, the CEWG serves as compensation for the individual in return for bearing risk of forced unemployment. The certainty equivalent wealth gain (CEWG) is computed as follows:

**Definition 4.2** $\Delta(x)$ *is called the certainty equivalent wealth gain (CEWG) at initial wealth* $x$ *if it satisfies*

$$\Phi(x - \Delta(x); I_1, 0) = \Phi(x; I_1, \delta),$$

*where* $\Phi(x; I_1, \delta)$ *is the value function described in (6) provided that the annual rate of income prior to retirement is* $I_1$ *and the unemployment intensity is* $\delta$*.*

CEWG increases as $\delta$ increases (Figure 7). Even for a small unemployment intensity $\delta = 0.5\%$, the CEWG can have substantial values of up to 10% of wealth; this disproportionately high reaction implies that an individual is willing to pay a substantial portion of her wealth to eliminate the unemployment risks. This observation is consistent with the fact that individuals (especially the poor) are significantly anxious about the presence of risk of forced unemployment.

**[Insert Figure 7 here.]**

CEWG increases as $\mu$ decreases or $\sigma$ increases (or both), because increase in an individual's investment opportunity decreases the amount of wealth that the individual is willing to pay to eliminate unemployment risks. (Figure 8).

**[Insert Figure 8 here.]**

CEWG is sensitive to changes of post-retirement leisure preference $K$ (Figure 9). As an individual's leisure demand $K$ increases, the CEWG required to buffer against unemployment risks increases, because the stress that the individual experiences due to the risk of forced unemployment increases as $K$ increases.

**[Insert Figure 9 here.]**

### 4.5 Unemployment Risks and Private Unemployment Insurance

The retirement problem considered in the previous sections is an optimal consumption and portfolio selection problem for an individual who wants to retire someday in the future, but who is exposed to a risk of forced unemployment, and is subject to borrowing constraints. Because we have assumed that unemployment risks cannot be fully diversified away, the financial market considered until now has been incomplete. In this section, we suggest a market innovation in the incomplete market by introducing the private unemployment insurance proposed by Jang *et al.* (2013), and demonstrate that providing private unemployment insurance in an incomplete market is beneficial to poor people and for people with a low post-retirement leisure preference, and that the insurance can be privately priced and be sold by private insurance providers.

We compute *reservation purchase price (RPP)* of private unemployment insurance and can say that the insurance is marketable at an equivalent or lower price than the RPP if insurance companies can successfully eliminate the moral hazard problem of the policy holders. The RPP is defined as the maximal lump-sum upfront insurance premium that the individual who is exposed to risk of forced unemployment is willing to pay to obtain the coverage of private unemployment insurance. We also compute *individual welfare benefit (IWB)* of the market innovation. The IWB is defined as the maximum wealth that an individual is willing to give up to eliminate her risk of forced unemployment by purchasing the private unemployment insurance. Our model confirms a positive IWB, and shows that utility can be gained by introducing private unemployment insurance.

To obtain empirically plausible implications, we match an individual's wealth-to-income ratios in our model (in which we fix income rate to one) to the ratios between family net worth and before-tax family income of the Survey of Consumer Finance (SCF). Following the SCF, we group U.S. families into percentiles of net worth during the period of 1995-2010 (Figure 10). Both family net worth and before-tax family income increase with percentile of net worth.

#### 4.5.1 Reservation Purchase Price of Unemployment Insurance

According to Jang *et al.* (2013), the actuarially-fair premium rate of the unemployment insurance is $\delta\big(\overline{X}(t) - X(t)\big)$, and the individual should pay it continuously if she wants insurance coverage of $\big(\overline{X}(t) - X(t)\big)$ when forced unemployment occurs.

We define a new value function for the case in which the individual enters the unemployment coverage without any insurance premium payment,

$$\widetilde{\Psi}(x) \equiv \max_{(c,\pi,\tau)} E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}\ln c(t)dt + e^{-\beta(\tau\wedge\tau_U)}\int_{\tau\wedge\tau_U}^{\infty} e^{-\beta(t-\tau\wedge\tau_U)}\ln(Kc(t))dt\Big], \tag{18}$$

which is subject to the wealth equation (4) and the borrowing constraints (5). We assume that wealth process is discontinuous at forced retirement time $\tau_U$, and consequently, the wealth $X(\tau_U)$ jumps to the critical wealth level $\overline{X}^*(\tau_U)$, described in (23), of the complete market case.[21] Following Damgaard (2003) we can define the RPP of the unemployment insurance as follows.[22]

**Definition 4.3** *The reservation purchase price (RPP) of the unemployment insurance is defined as* $\epsilon(x)$ *which satisfies*

$$\widetilde{\Psi}(x-\epsilon(x)) = \Phi(x),$$

*where* $\widetilde{\Psi}(x)$ *is the value function described in (18) and* $\Phi(x)$ *is the value function described in (6).*

The RPP also can be interpreted as the sum of subjectively-discounted unemployment insurance premiums paid by the individual policyholder.

An individual's RPP increases as her net worth decreases; this result implies that a person with low net worth will pay much more to obtain insurance coverage than will a person with high net worth (Table 3). Because poor individual's time to voluntary retirement is expected to be quite long, she has sufficient motivation to increase the amount of insurance that she purchases to hedge against the risk of forced unemployment. For example, an individual in the $0 \sim 25$ net worth group is willing to spend 59.42% of her wealth to purchase the unemployment insurance when $\mu = 0.1123$, $\sigma = 0.1954$, and $K = 3$. Moreover, as post-retirement leisure preference $K$ decreases, the price that an individual is willing to pay for insurance increases. An individual with a low post-retirement leisure preference tends to target a high (voluntary) retirement wealth level, so that she is likely to be exposed to

[21]The value function $\widetilde{\Psi}(x)$ is just a little variation of $\Psi(x)$. Therefore we can easily get the value function in a similar way.

[22]Damgaard (2003) provides a utility-based option valuation method in the presence of proportional transaction costs, and defines the reservation purchase and write price of a European call option. He shows that the reservation prices are the same as the option prices given by the classical Black-Scholes (1973) model if the financial market is complete and frictionless.

unemployment risks. This increases her motivation to purchase the insurance. We also find that a good investment opportunity reduces the RPP. Obviously, good market conditions reduce the stress that individuals feel due to risk of forced unemployment, because individuals are likely to reach voluntary retirement wealth level easily.

**[Insert Table 3 here.]**

### 4.5.2 Individual Welfare Benefit of Market Innovation

Many researchers have studied the IWB of unemployment insurance provided by government (Sheshinski and Weiss, 1979; Hamermesh, 1982; Abel, 1985; Kotlikoff *et al.*, 1987; Hansen and Imrohoroğlu, 1992; Acemoglu and Shimer, 1999), but none has explored the IWB of adopting private unemployment insurance. In this paper, we compute the IWB by comparing the value functions in the incomplete financial market described in the previous sections and in a corresponding complete market that is constructed by using private unemployment insurance (Section 2.3).

**Definition 4.4** *The individual welfare benefit (IWB) of private unemployment insurance is measured by* $\nabla(x)$ *satisfying*

$$\Psi(x - \nabla(x)) = \Phi(x),$$

*where* $\Psi(x)$ *is the value function described in (8), subject to the wealth equation (7) and the borrowing constraints (5), and* $\Phi(x)$ *is the value function described in (6).*

The IWB decreases with an individual's net worth; this trend implies that the introduction of private unemployment insurance might increase drastically the welfare of the poor (Table 6.8.4). Specifically, when $\mu = 0.1123$, $\sigma = 0.1954$, and $K = 3$, individuals in the $0 \sim 25$ net worth group get 17.10% of their wealth as the IWB. Moreover, the effect of market innovation becomes remarkable for individuals under a bad investment opportunity. As an extreme case when $\mu = 0.1023$, individuals in the $0 \sim 25$ net worth group get 24.89% of their wealth as the IWB. Moreover, an individual's IWB seems to increase as her post-retirement leisure preference $K$ decreases.

**[Insert Table 6.8.4 here.]**

## 5 Conclusion

We present a model of consumption, retirement, and asset allocation in an incomplete market; the model considers the case in which an individual is subject to risk of involuntary permanent unemployment that reduces her income severely, and who has borrowing constraints. Integrating flexibility in retirement timing and risks to labor income, we derive important implications for the relationship between borrowing constraints and an individual's retirement behaviors with unemployment risks in a utility maximizing framework. Results of this paper may be relevant to questions of policy regarding pension, insurance, and retirement.

Using with carefully-chosen parameters, our model produces results that are similar to those previously obtained in the literature on life-cycle consumption and portfolio with optimal retirement timing. We show that the interactions among consumption and portfolio choice can induce early retirement even when forced unemployment risks and borrowing constraints are considered jointly. More specifically, we find that retirement flexibility reduces consumption and increases in the risky portfolio, and that borrowing constraints and labor income risks induce early retirement.

This paper raises four main questions that should be considered in future research on life-cycle consumption and portfolio choice. First, in our modeling we adopted the certainly-restrictive assumption that the investment opportunity is constant: i.e., risk-free interest rate, expected rate of stock return, and stock volatility are all constant. Investigating the effects of a stochastic investment opportunity on an individual's optimal policies would be an interesting extension of this paper. To reflect a stochastic investment opportunity using the simplest possible setup, one can introduce a continuous-time Markov regime-switching model through which the effects of economic recessions and economic expansions on an individual's optimal strategies can be investigated.

Second, concerning the private unemployment insurance suggested by the paper, we can consider an insurance company that cannot distinguish whether or not the permanently unemployed are involuntarily retired to account for moral hazard. The moral hazard problem is very costly for the insurance company. Accordingly, an interesting open problem is to use an economy in which the insurance company cannot distinguish voluntary retirement from involuntary retirement.

Third, under the normative focus on the paper that investigates how the joint consideration of unemployment risks and borrowing constraints affects an individual's optimal strategies, we have ignored the positive equilibrium implications. Most importantly, a detailed equilibrium analysis and extensions of the paper would attempt to derive the general equilibrium implications for equity premium and risk-free interest rate in a Lucas-style equilibrium asset pricing model.

Finally, inclusion of the possibility that the individual can reenter the workforce after forced unemployment event would increase the realism of the model. Unemployed people should spend a job-searching cost to find a new job and can receive smaller labor income after reentering the workforce than before unemployment.

# 6 Appendix

## 6.1 Proof of Lemma 3.1

Recall that the value function $\Phi(x)$ is given by

$$\Phi(x) \equiv \max_{(c,\pi,\tau)} E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}\ln c(t)dt + e^{-\beta(\tau\wedge\tau_U)}\int_{\tau\wedge\tau_U}^{\infty} e^{-\beta(t-\tau\wedge\tau_U)}\ln\big(Kc(t)\big)dt\Big].$$

We define the value function of an individual who receives income at the rate equal to $I_2$ infinitely as the following:

$$U_2\big(X(\tau\wedge\tau_U)\big) \equiv \max_{(c,\pi)} E\Big[\int_{\tau\wedge\tau_U}^{\infty} e^{-\beta(t-\tau\wedge\tau_U)}\ln\big(Kc(t)\big)dt\Big].$$

Let $s = \tau\wedge\tau_U$. Then

$$U_2\big(X(s)\big) = \max_{(c,\pi)} E\Big[\int_{s}^{\infty} e^{-\beta(t-s)}\ln\big(Kc(t)\big)dt\Big].$$

Using the dynamic programming approach in Merton (1969, 1971) or the martingale approach in Karatzas and Shreve (1998), we obtain

$$U_2\big(X(s)\big) = \frac{1}{\beta}\Big[\ln\Big\{\beta\Big(X(s)+\frac{I_2}{r}\Big)\Big\} + \frac{1}{\beta}\Big(r+\frac{\theta^2}{2}-\beta(1-\ln K)\Big)\Big].$$

Further, by the principle of dynamic programming we can rewrite the value function $\Phi(x)$ given in (6) as the following:

$$\Phi(x) = \max_{(c,\pi,\tau)} E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}\ln c(t)dt + e^{-\beta(\tau\wedge\tau_U)}U_2\big(X(\tau\wedge\tau_U)\big)\Big].$$

The conditional expectation of $\tau_U$ allows us to obtain the following result:

$$
\begin{aligned}
&E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}\ln c(t)dt + e^{-\beta(\tau\wedge\tau_U)}U_2\big(X(\tau\wedge\tau_U)\big)\Big]\\
&= E\Big[E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}\ln c(t)dt + e^{-\beta(\tau\wedge\tau_U)}U_2\big(X(\tau\wedge\tau_U)\big)\Big]\Big|\tau_U\Big]\\
&= E\Big[\int_0^{\infty}\delta e^{-\delta s}\int_0^{\tau\wedge s} e^{-\beta t}\ln c(t)dtds + \int_0^{\infty}\delta e^{-\delta s}e^{-\beta(\tau\wedge s)}U_2\big(X(\tau\wedge s)\big)ds\Big]\\
&= E\Big[\int_0^{\tau}\delta e^{-\delta s}\int_0^{s} e^{-\beta t}\ln\{c(t)\}dtds + \int_\tau^{\infty}\delta e^{-\delta s}\int_0^{\tau} e^{-\beta t}\ln\{c(t)\}dtds\\
&\quad + \int_0^{\tau}\delta e^{-\delta s}e^{-\beta s}U_2\big(X(s)\big)ds + \int_\tau^{\infty}\delta e^{-\delta s}e^{-\beta\tau}U_2\big(X(\tau)\big)ds\Big]\\
&= E\Big[\int_0^{\tau} e^{-\beta t}\ln c(t)\int_t^{\tau}\delta e^{-\delta s}dsdt + \int_0^{\tau} e^{-\beta t}\ln c(t)\int_\tau^{\infty}\delta e^{-\delta s}dsdt\\
&\quad + \int_0^{\tau} e^{-(\beta+\delta)s}\delta U_2\big(X(s)\big)ds + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big]\\
&= E\Big[\int_0^{\tau} e^{-\beta t}\ln c(t)\int_t^{\infty}\delta e^{-\delta s}dsdt + \int_0^{\tau} e^{-(\beta+\delta)s}\delta U_2\big(X(s)\big)ds + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big]\\
&= E\Big[\int_0^{\tau} e^{-(\beta+\delta)t}\Big\{\ln c(t) + \delta U_2\big(X(t)\big)\Big\}dt + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big].
\end{aligned}
$$

## 6.2 Proof of Lemma 3.2

Our optimal retirement problem is the optimal retirement problem formulated by

$$\Phi(x) \equiv \max_{\tau} J_\tau(x),$$

where

$$J_\tau(x) \equiv \max_{(c,\pi)} E\Big[\int_0^{\tau} e^{-(\beta+\delta)t}\Big\{\ln c(t) + \delta U_2\big(X(t)\big)\Big\}dt + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big],$$

for a fixed stopping time $\tau$. We denote $c^*(t)$ and $\pi^*(t)$ by optimal consumption and risky portfolio strategies, respectively. We define the partial differential operator $L$ as follows:

$$L = \frac{\partial}{\partial t} + \Big(rx - c^*(t) + I_1 + \pi^*(t)\sigma\theta\Big)\frac{\partial}{\partial x} + \frac{1}{2}\pi^*(t)^2\sigma^2\frac{\partial^2}{\partial x^2}.$$

We define the domains $G$ and $D$ as the following:

$$G = \{(x,t)\in\mathbf{R}\times\mathbf{R}; x\geq 0, t\geq 0\}$$

and

$$D = \{(x,t)\in G; \tilde{\phi}(x,t) > e^{-(\beta+\delta)t}U_2(x)\}$$

for a function $\tilde{\phi} : \overline{G} \to \mathbf{R}$. Then the following relationship holds:

$$L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{ \ln c^*(t) + \delta U_2(x)\Big\} = \frac{\partial\tilde{\phi}}{\partial t} + \Big(rx - c^*(t) + I_1 + \pi^*(t)\sigma\theta\Big)\frac{\partial\tilde{\phi}}{\partial x} + \frac{1}{2}\pi^*(t)^2\sigma^2\frac{\partial^2\tilde{\phi}}{\partial x^2} + e^{-(\beta+\delta)t}\Big\{ \ln c^*(t) + \delta U_2(x)\Big\}.$$

Now we derive variational inequalities given in (10). Following Bensoussan and Lions (1982), and Øksendal (2007), the function $\tilde{\phi}$ satisfies the following variational inequalities:

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{ \ln c^*(t) + \delta U_2(x)\Big\} &= 0 \quad \text{on} \quad D, \\ L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{ \ln c^*(t) + \delta U_2(x)\Big\} &\le 0 \quad \text{on} \quad G\backslash D. \end{aligned}$$

As a result, the above variational inequalities are equivalent to the following:

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{ \ln c^*(t) + \delta U_2(x)\Big\} &\le 0, \\ \tilde{\phi}(x,t) &\ge e^{-(\beta+\delta)t}U_2(x), \\ \Big[L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{ \ln c^*(t) + \delta U_2(x)\Big\}\Big]\Big(\tilde{\phi}(x,t) - e^{-(\beta+\delta)t}U_2(x)\Big). \end{aligned} \tag{19}$$

We conjecture the form of $\tilde{\phi}$ as

$$\tilde{\phi}(x,t) = e^{-(\beta+\delta)t}\phi(x).$$

Substituting the above conjectured $\tilde{\phi}$ into the inequalities given in (19), we get

$$\begin{aligned} &\Big[-(\beta+\delta)\phi(x) + \Big(rx - c^*(t) + I_1 + \pi^*(t)\sigma\theta\Big)\phi'(x) \\ &\qquad + \frac{1}{2}\pi^*(t)^2\sigma^2\phi''(x) + \ln c^*(t) + \delta U_2(x)\Big] \le 0, \\ &\phi(x) \ge U_2(x), \\ &\Big[-(\beta+\delta)\phi(x) + \Big(rx - c^*(t) + I_1 + \pi^*(t)\sigma\theta\Big)\phi'(x) \\ &\qquad + \frac{1}{2}\pi^*(t)^2\sigma^2\phi''(x) + \ln c^*(t) + \delta U_2(x)\Big]\Big(\phi(x) - U_2(x)\Big) = 0. \end{aligned}$$

Note that optimality conditions for consumption and risky portfolio are given by

$$c^*(t) = \frac{1}{\phi'(x)} \quad \text{and} \quad \pi^*(t) = -\frac{\theta}{\sigma}\frac{\phi'(x)}{\phi''(x)},$$

which completes the proof of Lemma 3.2.

## 6.3 Proof of Lemma 3.3

We introduce a dual variable $\lambda$, which is defined as the marginal value of the value function $\phi(x)$. Further, the critical wealth level $\hat{x}$ over which it is optimal for an individual to enter voluntary retirement has an inverse relation with the dual variable $\lambda$. More specifically,

$$\lambda(x) \equiv \phi'(x) \quad \text{and} \quad \underline{\lambda} \equiv \frac{1}{\beta} \frac{1}{\hat{x} + I_2/r}.$$

We recall the nonlinear differential equation given in (11):

$$(\beta + \delta)\phi(x) - (rx + I_1)\phi'(x) + \frac{\theta^2}{2} \frac{\phi'(x)^2}{\phi''(x)} + 1 + \ln \phi'(x) = \delta U_2(x),$$

for any initial wealth $0 \le x < \hat{x}$. Differentiating the above equation with respect to $x$ yields

$$\lambda(x)(\theta^2 + \beta + \delta - r) + \frac{\lambda'(x)}{\lambda(x)} - \lambda'(x)(rx + I_1) - \frac{1}{2}\theta^2 \lambda(x)^2 \frac{\lambda''(x)}{\lambda'(x)^2} = \frac{\delta}{\beta} \frac{1}{x + I_2/r}. \tag{20}$$

We also introduce a function $G$ that is called the convex-dual function of the value function $\phi$:

$$G\big(\lambda(x)\big) \equiv x + \frac{I_1}{r},$$

which yields the following relationships:

$$G'\big(\lambda(x)\big)\lambda'(x) = 1 \quad \text{and} \quad G''\big(\lambda(x)\big)\lambda'(x)^2 + G'\big(\lambda(x)\big)\lambda''(x) = 0.$$

For the notational simplicity, we let $G\big(\lambda(x)\big) = G$ and $\lambda(x) = \lambda$. Then the (20) is written by using the convex-dual function $G$: for any $\underline{\lambda} < \lambda < \bar{\lambda}$

$$\lambda(\theta^2 + \beta + \delta - r) + \frac{1}{\lambda} \frac{1}{G'} - \frac{rG}{G'} + \frac{1}{2}\theta^2 \lambda^2 \frac{G''}{G'} = \frac{\delta}{\beta} \frac{1}{G - I_1/r - I_2/r}.$$

Rearranging this, we obtain the nonlinear equation given by (12) in Lemma 3.3.

## 6.4 Proof of Theorem 3.1

We can always write the general solution $G$ to the nonlinear equation (12) as follows:

$$G(\lambda) = \frac{1}{\lambda(\beta + \delta)} + A(\lambda)\lambda^{-\alpha_\delta} + A^*(\lambda)\lambda^{-\alpha_\delta^*}, \tag{21}$$

subject to

$$A'(\lambda)\lambda^{-\alpha_\delta} + (A^*(\lambda))'\lambda^{-\alpha_\delta^*} = 0.$$

The first and second derivatives of $G$ are given by

$$G'(\lambda) = -\frac{1}{\lambda^2}\frac{1}{\beta+\delta} - \alpha_\delta A(\lambda)\lambda^{-\alpha_\delta-1} - \alpha_\delta^* A^*(\lambda)\lambda^{-\alpha_\delta^*-1}$$

and

$$\begin{aligned}G''(\lambda) =& \frac{2}{\lambda^3}\frac{1}{\beta+\delta} + \alpha_\delta(\alpha_\delta+1)\lambda^{-\alpha_\delta-2}A(\lambda) + \alpha_\delta^*(\alpha_\delta^*+1)\lambda^{-\alpha_\delta^*-2}A^*(\lambda)\\ &- \alpha_\delta\lambda^{-\alpha_\delta-1}A'(\lambda) - \alpha_\delta^*\lambda^{-\alpha_\delta^*-1}(A^*(\lambda))',\end{aligned}$$

respectively. By using the general solution $G$ and its first and second derivatives, we get the following relationship:

$$\begin{aligned}&-\frac{1}{2}\theta^2\lambda^2 G''(\lambda) - \lambda G'(\lambda)(\theta^2+\beta+\delta-r) + rG(\lambda)\\ &= \frac{1}{\lambda} + \frac{\theta^2}{2}\Big(\alpha_\delta - \alpha_\delta^*\Big)\lambda^{1-\alpha_\delta}A'(\lambda).\end{aligned}$$

Using this we show that from the nonlinear equation (12)

$$\frac{\theta^2}{2}(\alpha_\delta - \alpha_\delta^*)\lambda^{1-\alpha_\delta}A'(\lambda) = -\frac{\delta}{\beta}\frac{G'(\lambda)}{G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}}$$

and

$$\frac{\theta^2}{2}(\alpha_\delta - \alpha_\delta^*)\lambda^{1-\alpha_\delta^*}(A^*(\lambda))' = \frac{\delta}{\beta}\frac{G'(\lambda)}{G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}}.$$

Then we get

$$A(\lambda) = A(\underline{\lambda}) - \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\int_{\underline{\lambda}}^{\lambda}\frac{\mu^{\alpha_\delta-1}G'(\mu)}{G(\mu) - \frac{I_1}{r} - \frac{I_2}{r}}d\mu$$

and

$$A^*(\lambda) = A^*(\bar{\lambda}) - \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\int_{\lambda}^{\bar{\lambda}}\frac{\mu^{\alpha_\delta^*-1}G'(\mu)}{G(\mu) - \frac{I_1}{r} - \frac{I_2}{r}}d\mu.$$

Hence, the general solution $G$ given by (21) can be restated as

$$\begin{aligned}G(\lambda) =& \frac{1}{\lambda(\beta+\delta)} + A(\underline{\lambda})\lambda^{-\alpha_\delta} + A^*(\bar{\lambda})\lambda^{-\alpha_\delta^*} - \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\Big[\lambda^{-\alpha_\delta}\int_{\underline{\lambda}}^{\lambda}\frac{\mu^{\alpha_\delta-1}G'(\mu)}{G(\mu) - \frac{I_1}{r} - \frac{I_2}{r}}d\mu\\ &+ \lambda^{-\alpha_\delta^*}\int_{\lambda}^{\bar{\lambda}}\frac{\mu^{\alpha_\delta^*-1}G'(\mu)}{G(\mu) - \frac{I_1}{r} - \frac{I_2}{r}}d\mu\Big].\end{aligned} \tag{22}$$

Using the following relationship

$$\frac{G'(\mu)}{G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}} = \frac{d}{d\mu}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\},$$

we can rewrite the general solution (22) as follows:

$$
\begin{aligned}
G(\lambda) = {} & \frac{1}{\lambda(\beta+\delta)} + A(\underline{\lambda})\lambda^{-\alpha_\delta} + A^*(\bar{\lambda})\lambda^{-\alpha_\delta^*} \\
& - \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[\lambda^{-\alpha_\delta}\underline{\lambda}^{\alpha_\delta-1}\ln\Big\{\beta\Big(G(\underline{\lambda})-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\} - \lambda^{-\alpha_\delta}\underline{\lambda}^{\alpha_\delta-1}\ln\frac{1}{\underline{\lambda}} \\
& - \lambda^{-\alpha_\delta}\int_{\underline{\lambda}}^{\lambda}(\alpha_\delta-1)\mu^{\alpha_\delta-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu \\
& + \lambda^{-\alpha_\delta^*}\bar{\lambda}^{\alpha_\delta^*-1}\ln\Big(\beta\frac{I_2}{r}\Big) - \lambda^{-\alpha_\delta^*}\bar{\lambda}^{\alpha_\delta^*-1}\ln\Big\{\beta\Big(G(\bar{\lambda})-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu \\
& - \lambda^{-\alpha_\delta^*}\int_{\lambda}^{\bar{\lambda}}(\alpha_\delta^*-1)\mu^{\alpha_\delta^*-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu\Big].
\end{aligned}
$$

When we define

$$
\begin{aligned}
B(\underline{\lambda}) &\equiv A(\underline{\lambda}) - \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)}\underline{\lambda}^{\alpha_\delta-1}\ln\Big\{\beta\Big(G(\underline{\lambda})-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\} + \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\underline{\lambda}^{\alpha_\delta-1}\ln\frac{1}{\underline{\lambda}}, \\
B^*(\bar{\lambda}) &\equiv A^*(\bar{\lambda}) - \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\bar{\lambda}^{\alpha_\delta^*-1}\ln\Big(\beta\frac{I_2}{r}\Big) + \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\bar{\lambda}^{\alpha_\delta^*-1}\ln\Big\{\beta\Big(G(\bar{\lambda})-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\},
\end{aligned}
$$

we obtain the analytic solution $G$ proposed by (13).

### 6.5 Proof of Theorem 3.2

When we derive the variational inequality (10), we have used the following optimality conditions for consumption and risky investment:

$$
c^*(t) = \frac{1}{\phi'(x)} \quad \text{and} \quad \pi^*(t) = -\frac{\theta}{\sigma}\frac{\phi'(x)}{\phi''(x)}.
$$

Recall the following relations between the marginal value $\phi'(x)$ of the value function $\phi(x)$ and the convex-dual function $G(\lambda)$:

$$
\lambda(x) = \phi'(x) \quad \text{and} \quad G\big(\lambda(x)\big) = x + \frac{I_1}{r}.
$$

Using these relations we rewrite the optimality conditions in terms of $G(\lambda)$

$$
c^*(t) = \frac{1}{\lambda^*(x)} \quad \text{and} \quad \pi^*(t) = -\frac{\theta}{\sigma}\lambda^*(x)G'\big(\lambda^*(x)\big).
$$

Also, utilizing the general solution $G$ given by (13) yields

$$
\begin{aligned}
x + \frac{I_1}{r} = G\big(\lambda^*(x)\big) = {} & \frac{1}{\lambda^*(x)(\beta+\delta)} + B(\underline{\lambda})\lambda^*(x)^{-\alpha_\delta} + B(\bar{\lambda})\lambda^*(x)^{-\alpha_\delta^*} \\
& + \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_{\underline{\lambda}}^{\lambda^*(x)}\mu^{\alpha_\delta-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu \\
& + (\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\bar{\lambda}}\mu^{\alpha_\delta^*-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu\Big].
\end{aligned}
$$

Rearranging this gives the optimal consumption $c^*(t)$ given in Theorem 3.2. A direct calculation of the first derivative of $G$ gives

$$G'(\lambda^*(x)) = -\frac{1}{\lambda^*(x)^2(\beta+\delta)} - \alpha_\delta B(\underline{\lambda})\lambda^*(x)^{-\alpha_\delta-1} - \alpha_\delta^* B^*(\bar{\lambda})\lambda^*(x)^{-\alpha_\delta^*-1} + \frac{2\delta}{\theta^2\beta}\ln\Big\{\beta\Big(G(\lambda^*(x)) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}$$
$$- \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[\alpha_\delta(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta-1}\int_{\underline{\lambda}}^{\lambda^*(x)}\mu^{\alpha_\delta-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu$$
$$+ \alpha_\delta^*(\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*-1}\int_{\lambda^*(x)}^{\bar{\lambda}}\mu^{\alpha_\delta-2}\ln\Big\{\beta\Big(G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}\Big)\Big\}d\mu\Big].$$

Substituting this into the optimality condition $\pi^*(t)$ completes the proof of Theorem 3.2.

Note that we can get an upper bound of the optimal risky asset holdings when the individual is about to retire and a lower bound when she becomes extremely poor:

$$\lim_{x\uparrow\hat{x}}\pi(t) \le \frac{\theta}{\sigma}\frac{1}{\underline{\lambda}(\beta+\delta)} + \frac{\theta}{\sigma}\alpha_\delta B(\underline{\lambda})\underline{\lambda}^{-\alpha_\delta} + \frac{\theta}{\sigma}\alpha_\delta^* B^*(\bar{\lambda})\underline{\lambda}^{-\alpha_\delta^*} + \frac{2\delta\ln\underline{\lambda}}{\sigma\theta\beta\underline{\lambda}}\Big(\frac{\alpha_\delta}{\alpha_\delta-\alpha_\delta^*}\Big)$$
$$-\frac{2\delta\alpha_\delta^*\ln\underline{\lambda}}{\sigma\theta\beta(\alpha_\delta-\alpha_\delta^*)}\underline{\lambda}^{-\alpha_\delta^*}\bar{\lambda}^{\alpha_\delta^*-1}.$$

Moreover,

$$\lim_{x\downarrow 0}\pi(t) \ge \frac{\theta}{\sigma}\frac{1}{\bar{\lambda}(\beta+\delta)} + \frac{\theta}{\sigma}\alpha_\delta B(\underline{\lambda})\bar{\lambda}^{-\alpha_\delta} + \frac{\theta}{\sigma}\alpha_\delta^* B^*(\bar{\lambda})\bar{\lambda}^{-\alpha_\delta^*} + \frac{2\delta}{\sigma\theta\beta\bar{\lambda}}\ln\Big(\frac{\beta I_2}{r}\Big)\frac{\alpha_\delta^*}{\alpha_\delta-\alpha_\delta^*}$$
$$-\frac{2\delta\alpha_\delta}{\sigma\theta\beta(\alpha_\delta-\alpha_\delta^*)}\ln\Big(\frac{\beta I_2}{r}\Big)\bar{\lambda}^{-\alpha_\delta}\underline{\lambda}^{\alpha_\delta-1}.$$

To verify this, we first use the inequality

$$G(\mu) - \frac{I_1}{r} + \frac{I_2}{r} \le \frac{1}{\beta\underline{\lambda}},$$

and take the limit of $\lambda^* \downarrow \underline{\lambda}$ to derive the last term of the right hand side of the first inequality. The last term of the right hand side of the second inequality can be derived if we use $G(\mu) \ge 0$ for all $\mu$ and take the limit of $x \downarrow 0$.

### 6.6 The Details of Deriving the Value Function $\Psi(x)$

We know that

$$\Psi(x) = \max_{\tau^c}\psi_{\tau^c}(x),$$

where $\psi_{\tau^c}$ is defined as

$$\psi_{\tau^c}(x) \equiv \max_{(c,\pi,\overline{X})} E\Big[\int_0^{\tau^c} e^{-(\beta+\delta)t}\Big\{\ln c(t) + \delta U_2\big(\overline{X}(t)\big)\Big\}dt + e^{-(\beta+\delta)\tau^c}U_2\big(X(\tau^c)\big)\Big].$$

The optimal stopping problem $\Psi(x) = \max_{\tau^c} \psi_{\tau^c}(x)$ is equivalent to solving the variational inequality (Bensoussan and Lions, 1982; Øksendal, 2007)

$$(\beta+\delta)\varphi(x) - (rx + I_1 + \delta\frac{I_2}{r})\varphi'(x) + \frac{\theta^2}{2}\frac{\varphi'(x)^2}{\varphi''(x)} + 1 + \frac{\delta}{\beta} + (1+\frac{\delta}{\beta})\ln\varphi'(x)$$
$$- \frac{\delta}{\beta^2}(r + \frac{\theta^2}{2} - \beta(1-\ln K)) \geq 0,$$
$$\varphi(x) \geq U_2(x).$$

Here, we utilized the optimality conditions with respect to $c$, $\pi$, and $\overline{X}$, respectively:

$$c^* = \frac{1}{\varphi'(x)}, \quad \pi^* = -\frac{1}{x}\frac{\theta}{\sigma}\frac{\varphi'(x)}{\varphi''(x)}, \quad \overline{X}^* = \frac{1}{\beta\varphi'(x)} - \frac{I_2}{r}. \tag{23}$$

Thus, we construct a problem with one free boundary from the above inequality

$$\begin{cases} (\beta+\delta)\varphi(x) - (rx + I_1 + \delta\frac{I_2}{r})\varphi'(x) & +\frac{\theta^2}{2}\frac{\varphi'(x)^2}{\varphi''(x)} + 1 + \frac{\delta}{\beta} + (1+\frac{\delta}{\beta})\ln\varphi'(x) \\ & -\frac{\delta}{\beta^2}(r + \frac{\theta^2}{2} - \beta(1-\ln K)) = 0, \quad 0 \leq x < \hat{x}^c, \\ \varphi(x) = U_2(x), \quad x \geq \hat{x}^c, & \\ \varphi(\hat{x}) = U_2(\hat{x}^c), & \\ \varphi'(\hat{x}) = \frac{1}{\beta}\frac{1}{\hat{x}^c + I_2/r}, & \end{cases} \tag{24}$$

where $\hat{x}^c$ is the critical wealth level to be determined in a complete market innovated by introduction of private unemployment insurance. We use the conventional convex-duality method, and subsequently, use the convex-dual function $\widetilde{\varphi}(y)$ defined as

$$\varphi(x) = \inf_{y \in \mathbf{R}} \{\widetilde{\varphi}(y) + xy\}.$$

Then the free boundary problem (24) is rewritten as

$$\frac{\theta^2}{2}y^2\widetilde{\varphi}''(y) + \{\beta + \delta - r - \delta\}y\widetilde{\varphi}'(y) - (\beta+\delta)\widetilde{\varphi}(y) + C(y) = 0, \quad \text{for} \;\; \underline{y} < y < \overline{y}, \tag{25}$$

with boundary conditions with respect to two boundaries $\underline{y}$, $\overline{y}$

$$\begin{aligned} &\widetilde{\varphi}(\underline{y}) = -\frac{1}{\beta}\ln\underline{y} + \frac{I_2}{r}\underline{y} + \frac{1}{\beta}\Big\{\frac{1}{\beta}\Big(r + \frac{\theta^2}{2} - \beta(1-\ln K)\Big) - 1\Big\}, \\ &\widetilde{\varphi}'(\underline{y}) = -\frac{1}{\beta\underline{y}} + \frac{I_2}{r}, \\ &\widetilde{\varphi}'(\overline{y}) = 0, \\ &\widetilde{\varphi}''(\overline{y}) = 0, \end{aligned} \tag{26}$$

and $C(y)$ is defined as

$$C(y) \equiv \Big(I_1 + \frac{\delta I_2}{r}\Big)y - (1+\frac{\delta}{\beta})\ln y - 1 - \frac{\delta}{\beta} + \frac{\delta}{\beta^2}\{r + \frac{\theta^2}{2} - \beta(1-\ln K)\}.$$

The variation of parameter method yields the general solution to ODE (25)

$$\widetilde{\varphi}(y) = A_1 y^{n_1} + A_2 y^{n_2} + \widetilde{\varphi}_p(y),$$

where $A_1$, $A_2$ are constants to be determined, and

$$n_1 = \frac{-\big(\beta+\delta-r-\delta-\frac{\theta^2}{2}\big) + \sqrt{\big(\beta+\delta-r-\delta-\frac{\theta^2}{2}\big)^2 + 2(\beta+\delta)\theta^2}}{\theta^2},$$

$$n_2 = \frac{-\big(\beta+\delta-r-\delta-\frac{\theta^2}{2}\big) - \sqrt{\big(\beta+\delta-r-\delta-\frac{\theta^2}{2}\big)^2 + 2(\beta+\delta)\theta^2}}{\theta^2},$$

$$\begin{aligned}\widetilde{\varphi}_p(y) = {} & \Big(I_1 + \frac{\delta I_2}{r}\Big)\frac{1}{(r+\delta)}y - \frac{1}{\beta}\ln y - \frac{1}{\theta^2\beta}\Big(\frac{\beta+\delta}{r+\delta} - 1\Big) \\ & - \frac{1}{\beta(\beta+\delta)}\Big\{1 + \frac{\delta}{\beta} - \frac{\delta}{\beta}\Big(r + \frac{\theta^2}{2} - \beta(1-\ln K)\Big)\Big\}.\end{aligned}$$

From the boundary conditions in (26) we obtain two equations

$$\begin{aligned} & \Big[-\frac{n_2}{\beta}\ln \underline{y} + \frac{I_2}{r}\underline{y}(n_2-1) + \frac{n_2}{\beta}\Big\{\frac{1}{\beta}\Big(r+\frac{\theta^2}{2} - \beta(1-\ln K)\Big) - 1\Big\} + \frac{1}{\beta} - n_2\widetilde{\varphi}_p(\underline{y}) \\ & + \underline{y}\widetilde{\varphi}_p'(\underline{y})\Big] n_1(n_2-1)\overline{y}^{n_1-1} = \Big(-(n_2-1)\widetilde{\varphi}_p(\overline{y}) + \overline{y}\widetilde{\varphi}_p''(\overline{y})\Big)(n_2-n_1)\underline{y}^{n_1}, \\ & \Big[-\frac{n_1}{\beta}\ln \underline{y} + \frac{I_2}{r}\underline{y}(n_1-1) + \frac{n_1}{\beta}\Big\{\frac{1}{\beta}\Big(r+\frac{\theta^2}{2} - \beta(1-\ln K)\Big) - 1\Big\} + \frac{1}{\beta} - n_1\widetilde{\varphi}_p(\underline{y}) \\ & + \underline{y}\widetilde{\varphi}_p'(\underline{y})\Big] n_2(n_1-1)\overline{y}^{n_2-1} = \Big(-(n_1-1)\widetilde{\varphi}_p(\overline{y}) + \overline{y}\widetilde{\varphi}_p''(\overline{y})\Big)(n_1-n_2)\underline{y}^{n_2}. \end{aligned} \tag{27}$$

$\underline{y}$ and $\bar{y}$ are determined numerically, and then $A_1$ and $A_2$ are easily obtained from the boundary conditions in (26).

The optimality conditions (23) give optimal consumption $c^*$ and risky investment $\pi^*$. The critical wealth level $\hat{x}^c$ is derived from the relationship $\varphi'(\hat{x}) = \dfrac{1}{\beta}\dfrac{1}{\hat{x} + I_2/r}$ in (24).

## 6.7 The Iterative Method

The value-matching condition of $\phi(\hat{x}) = U_2(\hat{x})$ in the free boundary problem (11) yields

$$\ln K = \underline{\lambda}(I_1 - I_2)\Big(1 + \frac{\theta^2\alpha_\delta^*}{2r}\Big) + \frac{\delta}{\beta}\ln\underline{\lambda} + \frac{\theta^2(\alpha_\delta - \alpha_\delta^*)}{2}B(\underline{\lambda})\underline{\lambda}^{-\alpha_\delta+1}. \tag{28}$$

Firstly, we will show that the equation (28) can be followed by the value-matching condition at $\lambda = \underline{\lambda}$, or equivalently at $x = \hat{x}$. Rearranging the first equation in (11) we get an equality concerning $\phi(x)$

$$(\beta + \delta)\phi(x) = (rx + I_1)\lambda(x) - \frac{\theta^2}{2}\lambda^2(x)G'\big(\lambda(x)\big) - \big(1 + \ln\lambda(x)\big) + \delta U_2(x). \tag{29}$$

If we let

$$\begin{aligned} H(\lambda) \equiv \frac{1}{(\beta+\delta)}\Big[ & rG(\lambda)\lambda - \frac{\theta^2}{2}\lambda^2 G'(\lambda) - (1 + \ln\lambda) \\ & + \frac{\delta}{\beta}\Big\{ \ln\Big\{ G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r} \Big\} + \frac{1}{\beta}\Big( r + \frac{\theta^2}{2} - \beta(1 - \ln K)\Big)\Big\}\Big], \end{aligned} \tag{30}$$

then

$$\phi(x) = H\big(\lambda(x)\big). \tag{31}$$

Equations (12) and (30) yield

$$H'(\lambda) = \lambda G'(\lambda),$$

so that

$$\phi'(x) = H'\big(\lambda(x)\big)\lambda'(x) = \frac{H'(\lambda(x))}{G'(\lambda(x))} = \lambda(x).$$

Therefore, $\phi(x)$ is a solution of the first equation in (11) subject to a boundary condition

$$\phi'(\hat{x}) = \lambda(\hat{x}) = \underline{\lambda} = \frac{1}{\beta}\frac{1}{\hat{x} + \frac{I_2}{r}}.$$

Using the value-matching condition

$$\phi(\hat{x}) = U_2(\hat{x})$$

in (11), we obtain the value of $H$ at $\lambda = \hat{\lambda}$

$$H(\underline{\lambda}) = \frac{1}{\beta}\Big[ \ln\frac{1}{\underline{\lambda}} + \frac{1}{\beta}\Big( r + \frac{\theta^2}{2} - \beta(1 - \ln K)\Big)\Big]. \tag{32}$$

Therefore, we get

$$\begin{aligned} (\beta+\delta)\frac{1}{\beta}\Big[ \ln\frac{1}{\underline{\lambda}} + \frac{1}{\beta}\Big( r + \frac{\theta^2}{2} - \beta(1 - \ln K)\Big)\Big] &= r\underline{\lambda}\Big(\frac{1}{\beta\underline{\lambda}} + \frac{I_1 - I_2}{r}\Big) - \frac{\theta^2}{2}\underline{\lambda}^2 G'(\underline{\lambda}) \\ & - (1 + \ln\underline{\lambda}) + \frac{\delta}{\beta}\Big[ \ln\frac{1}{\underline{\lambda}} + \frac{1}{\beta}\Big( r + \frac{\theta^2}{2} - \beta(1 - \ln K)\Big)\Big] \end{aligned} \tag{33}$$

if we rearrange the relationship (30) and rewrite it at the boundary $\lambda = \underline{\lambda}$.

The smooth-pasting condition of $\phi'(\hat{x}) = \dfrac{1}{\beta}\dfrac{1}{\hat{x}+I_2/r}$ in the free boundary problem (11) can be rewritten by using the free boundary $\underline{\lambda}$:

$$\underline{\lambda} = \frac{1}{\beta}\frac{1}{\hat{x}+I_2/r},$$

accordingly

$$\hat{x} = \frac{1}{\beta\underline{\lambda}} - \frac{I_2}{r}.$$

Utilizing the general solution $G$ given by (13) we obtain

$$\hat{x} + \frac{I_1}{r} = G(\underline{\lambda}),$$

which can be restated as

$$\begin{aligned}\frac{1}{\beta\underline{\lambda}} + \frac{I_1 - I_2}{r} = {} & \frac{1}{\underline{\lambda}(\beta+\delta)} + B(\underline{\lambda})\underline{\lambda}^{-\alpha_\delta} + B^*(\bar{\lambda})\underline{\lambda}^{-\alpha_\delta^*} \\ & + \frac{2\delta(\alpha_\delta^* - 1)\underline{\lambda}^{-\alpha_\delta^*}}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\int_{\underline{\lambda}}^{\bar{\lambda}} \mu^{\alpha_\delta^*-2}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu.\end{aligned} \tag{34}$$

Now we use the value-matching and smooth-pasting conditions of the value function $\phi(x)$ at zero wealth level. The value-matching condition is given as follows: if we let $\bar{\lambda} = \phi'(0)$, then

$$G(\bar{\lambda}) = \frac{I_1}{r},$$

which can be rewritten by

$$\frac{I_1}{r} = \frac{1}{\bar{\lambda}(\beta+\delta)} + B(\underline{\lambda})\bar{\lambda}^{-\alpha_\delta} + B^*(\bar{\lambda})\bar{\lambda}^{-\alpha_\delta^*} + \frac{2\delta(\alpha_\delta - 1)\bar{\lambda}^{-\alpha_\delta}}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\int_{\underline{\lambda}}^{\bar{\lambda}} \mu^{\alpha_\delta-2}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu. \tag{35}$$

The smooth-pasting condition is given by the fact that if an individual's initial wealth approaches zero, then the individual optimally has zero risky investment position. Technically, the fact is represented by

$$G'(\bar{\lambda}) = 0,$$

because of the optimality condition for risky investment is given by

$$\pi^*(t) = -\frac{\theta}{\sigma}\lambda^*(x)G'\big(\lambda^*(x)\big).$$

As a result, we get

$$\begin{aligned}0 = {} & -\frac{1}{\bar{\lambda}^2(\beta+\delta)} - \alpha_\delta B(\underline{\lambda})\bar{\lambda}^{-\alpha_\delta-1} - \alpha_\delta^* B^*(\bar{\lambda})\bar{\lambda}^{-\alpha_\delta^*-1} + \frac{2\delta}{\theta^2\beta\bar{\lambda}^2}\ln\Big(\beta\frac{I_2}{r}\Big) \\ & - \frac{2\delta\alpha_\delta(\alpha_\delta - 1)\bar{\lambda}^{-\alpha_\delta-1}}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\int_{\underline{\lambda}}^{\bar{\lambda}} \mu^{\alpha_\delta-2}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu.\end{aligned} \tag{36}$$

Combining (34), (35), and (36), we can derive the equation (28).

A little rearrangement of second and third equality in (35) and (36), respectively, gives

$$\frac{\alpha_\delta - 1}{\beta + \delta} + \frac{2\delta}{\theta^2\beta}\ln\left(\beta\frac{I_2}{r}\right) = \frac{\alpha_\delta I_1}{r}\bar{\lambda} - (\alpha_\delta - \alpha_\delta^*)B^*(\bar{\lambda})\bar{\lambda}^{-\alpha_\delta^*+1}. \tag{37}$$

Then from the relationships of (28) and (37), we can rewrite $B(\underline{\lambda})$ and $B^*(\bar{\lambda})$ as functions of $\underline{\lambda}$ and $\bar{\lambda}$, respectively. By substituting $B(\underline{\lambda})$ and $B^*(\bar{\lambda})$ into the equations given in (34) and (35), we finally get the following two equations:

$$\begin{aligned}
&\frac{I_1 - I_2}{r} + (I_1 - I_2)\Big(1 + \frac{\theta^2\alpha_\delta^*}{2r}\Big)\frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)} \\
&\qquad = \frac{1}{\underline{\lambda}(\beta+\delta)} - \frac{1}{\beta\underline{\lambda}} + \Big\{\frac{\ln K}{\underline{\lambda}} - \frac{\delta\ln\underline{\lambda}}{\beta\underline{\lambda}}\Big\}\frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)} \\
&\qquad + \Big[\frac{\alpha_\delta I_1}{r\bar{\lambda}} - \Big\{\frac{\alpha_\delta - 1}{\beta+\delta} + \frac{2\delta}{\theta^2\beta}\ln\Big(\beta\frac{I_2}{r}\Big)\Big\}\frac{1}{\bar{\lambda}^2}\Big]\frac{1}{(\alpha_\delta - \alpha_\delta^*)}\Big(\frac{\bar{\lambda}}{\underline{\lambda}}\Big)^{\alpha_\delta^*}\bar{\lambda} \\
&\qquad + \frac{2\delta(\alpha_\delta^* - 1)\underline{\lambda}^{-\alpha_\delta^*}}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\int_{\underline{\lambda}}^{\bar{\lambda}}\mu^{\alpha_\delta^*-2}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu
\end{aligned} \tag{38}$$

$$\begin{aligned}
&-\frac{I_1}{r}\frac{\alpha_\delta^*}{(\alpha_\delta - \alpha_\delta^*)} \\
&\quad = \frac{1}{\bar{\lambda}(\beta+\delta)} + \Big\{\ln K - (I_1 - I_2)\Big(1 + \frac{\theta^2\alpha_\delta^*}{2r}\Big)\underline{\lambda} - \frac{\delta}{\beta}\ln\underline{\lambda}\Big\}\frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\Big(\frac{\underline{\lambda}}{\bar{\lambda}}\Big)^{\alpha_\delta}\frac{1}{\underline{\lambda}} \\
&\quad - \Big[\Big\{\frac{\alpha_\delta - 1}{\beta+\delta} + \frac{2\delta}{\theta^2\beta}\ln\Big(\beta\frac{I_2}{r}\Big)\Big\}\frac{1}{\bar{\lambda}}\Big]\frac{1}{(\alpha_\delta - \alpha_\delta^*)} + \frac{2\delta(\alpha_\delta - 1)\bar{\lambda}^{-\alpha_\delta}}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\int_{\underline{\lambda}}^{\bar{\lambda}}\mu^{\alpha_\delta-2}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu.
\end{aligned} \tag{39}$$

Our problem reduces to numerically determine two free boundaries $\underline{\lambda}$ and $\bar{\lambda}$ in (38) and (39).

Now we present an iterative method to show graphical illustration of our main results given in Theorem 3.2.

**The iterative procedure**

- (Step 0) Notice that, if $\delta = 0$, we easily get $B(\underline{\lambda})$ from (28) and $B^*(\bar{\lambda})$ from (37). Then we obtain $G(\lambda)$ from (13). Putting the $G(\lambda)$ into two equations in (38) and (38), we get $\underline{\lambda}$ and $\bar{\lambda}$. Suppose $\delta \neq 0$, but has a sufficiently small value.[23] We exploit $G(\lambda)$, $\underline{\lambda}$ and $\bar{\lambda}$ for the case in which $\delta = 0$ as the initial values of our iteration method.

[23]This condition of $\delta$ is necessary in order to guarantee the uniqueness and monotonicity of the solution of (13) and to verify the fact that the solution obtained from the new convex-dual approach is a solution of the free boundary problem (11). We display the possible range of $\delta$ in Theorem 6.1, 6.2 in Appendix.

- (Step 1) Since we have initial values $\underline{\lambda}$, $\bar{\lambda}$ and $G(\lambda)$, we get $B(\underline{\lambda})$ and $B^*(\bar{\lambda})$ from (28) and (37), respectively.
- (Step 2) Update $G(\lambda)$ by using the equation (13).
- (Step 3) Putting the updated $G(\lambda)$ into the equations in (38) and (39), we obtain new $\underline{\lambda}$ and $\bar{\lambda}$.
- (Step 4) Repeat steps 1, 2 and 3 until $\underline{\lambda}$ and $\bar{\lambda}$ converge.

## 6.8 Various Properties of Convex-Dual Function $G$

### 6.8.1 Uniqueness of the Solution of (13)

**Theorem 6.1** *(Uniqueness) If* $\dfrac{2\delta}{\theta^2\beta}\dfrac{(\bar{\lambda}-\underline{\lambda})}{\underline{\lambda}\,\bar{\lambda}} < 1$, *the solution of (13) is unique.*

**Proof.** Let $G_1$ and $G_2$ be the two solutions of (13), then we get

$$\begin{aligned}
G_1(\lambda) - G_2(\lambda) =& \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\Big[(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_{\underline{\lambda}}^{\lambda}\mu^{\alpha_\delta - 2}\Big(\ln\Big\{\beta\Big(G_1(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\} \\
&- \ln\Big\{\beta\Big(G_2(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}\Big)d\mu \\
&+ (\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\bar{\lambda}}\mu^{\alpha_\delta^* - 2}\Big(\ln\Big\{\beta\Big(G_1(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\} \\
&- \ln\Big\{\beta\Big(G_2(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}\Big)d\mu\Big].
\end{aligned}$$

We know that

$$\Big|\ln\Big\{\beta\Big(G_1(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\} - \ln\Big\{\beta\Big(G_2(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}\Big| \le |G_1(\mu) - G_2(\mu)|,$$

and it implies

$$|G_1(\lambda) - G_2(\lambda)| \le \frac{2\delta}{\theta^2\beta}\frac{(\bar{\lambda}-\underline{\lambda})}{\underline{\lambda}\,\bar{\lambda}}\sup_{\mu}|G_1(\mu) - G_2(\mu)|.$$

Hence, the proof is complete. **Q.E.D.**

### 6.8.2 The Strictly Decreasing Property of Convex-Dual Function $G$

The following theorem permits us to get a monotonically decreasing $G(\lambda)$ under suitable parameter conditions.

**Theorem 6.2** *(Monotonicity) Suppose that*

$$\begin{aligned}&\ln K > \underline{\lambda}(I_1 - I_2)\Big(1 + \frac{\theta^2\alpha_\delta^*}{2r}\Big) + \frac{\delta}{\beta}\ln\underline{\lambda},\\ &\frac{\alpha_\delta(\alpha_\delta^* - 1)}{(\alpha_\delta^* - \alpha_\delta)(\beta + \delta)} > \frac{2\delta}{\theta^2\beta}\Big\{\ln\Big(\frac{1}{\underline{\lambda}}\Big) - \frac{\alpha_\delta^*}{(\alpha_\delta^* - \alpha_\delta)}\ln\Big(\beta\frac{I_2}{r}\Big)\Big\} + \frac{\alpha_\delta\alpha_\delta^* I_1}{(\alpha_\delta^* - \alpha_\delta)r}\bar{\lambda},\end{aligned} \tag{40}$$

*Then, any solution of (13) satisfies $G'(\lambda) < 0$.*

**Proof.** Any solution of (12) satisfies the integral equation (13). From (28) and the first assumption of Theorem 6.2, we deduce

$$B(\underline{\lambda})\underline{\lambda}^{1-\alpha_\delta} = \Big[\ln K - \underline{\lambda}(I_1 - I_2)\Big(1 + \frac{\theta^2\alpha_\delta^*}{2r}\Big) - \frac{\delta}{\beta}\ln\underline{\lambda}\Big]\frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)} > 0.$$

We compute $G'(\lambda)$

$$\begin{aligned}G'(\lambda) = &-\frac{1}{\lambda^2(\beta+\delta)} - \alpha_\delta B(\underline{\lambda})\lambda^{-\alpha_\delta - 1} - \alpha_\delta^* B^*(\bar{\lambda})\lambda^{-\alpha_\delta^* - 1} + \frac{2\delta}{\theta^2\beta}\ln\Big\{\beta\Big(G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}\\ &- \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\Big[\alpha_\delta(\alpha_\delta - 1)\lambda^{-\alpha_\delta - 1}\int_{\underline{\lambda}}^{\lambda}\mu^{\alpha_\delta - 2}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu\\ &+ \alpha_\delta^*(\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^* - 1}\int_{\lambda}^{\bar{\lambda}}\mu^{\alpha_\delta - 2}\ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu\Big]\\ \le &-\alpha_\delta B(\underline{\lambda})\lambda^{-\alpha_\delta - 1} - \frac{1}{\lambda^2}\Big[\frac{\alpha_\delta(\alpha_\delta^* - 1)}{(\alpha_\delta^* - \alpha_\delta)(\beta+\delta)} - \frac{2\delta}{\theta^2\beta}\Big\{\ln\Big(\frac{1}{\underline{\lambda}}\Big) - \frac{\alpha_\delta^*}{(\alpha_\delta^* - \alpha_\delta)}\ln\Big(\beta\frac{I_2}{r}\Big)\Big\} - \frac{\alpha_\delta\alpha_\delta^* I_1}{(\alpha_\delta^* - \alpha_\delta)r}\bar{\lambda}\Big],\end{aligned}$$

where we have used the relationship (37) and the fact that

$$\frac{I_2}{r} \le G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r} \le \frac{1}{\beta\underline{\lambda}}, \quad \text{for } \underline{\lambda} \le \lambda \le \bar{\lambda}$$

to obtain the last inequality. Therefore, we conclude that $G'(\lambda) < 0$. **Q.E.D.**

### 6.8.3 The Equivalence between the Problem (11) and the Variational Inequality (10)

If we find $\phi(x)$ satisfying the variational inequality (10), then the $\phi(x)$ certainly is a solution to the free boundary problem (11). Hence, now we verify that the solution $\phi(x)$ of the problem (11) is a solution of the variational inequality (10).

**Theorem 6.3** *If we take the assumption of (40) and further assume that*

$$\underline{\lambda} \le \ln K/(I_1 - I_2), \quad \bar{\lambda} < \frac{r}{\beta I_2},$$

*then the solution $\phi(x)$ of the problem (11) is a solution of the variational inequality (10).*

**Proof.** Define

$$f(x) \equiv \phi(x) - U_2(x).$$

From $\phi(\hat{x}) = U_2(\hat{x})$, $f(\hat{x}) = 0$. If we show that $f'(x) \leq 0$ for $0 \leq x \leq \hat{x}$, then the second inequality in (10) will follow. To do this, it is enough to show that

$$G(\lambda) < \frac{1}{\beta\lambda} + \frac{I_1 - I_2}{r}, \tag{41}$$

for $\underline{\lambda} < \lambda < \bar{\lambda}$. Let

$$\Gamma(\lambda) \equiv G(\lambda) - \frac{1}{\beta\lambda} - \frac{I_1 - I_2}{r}.$$

Then $\Gamma(\underline{\lambda}) = 0$ because $G(\underline{\lambda}) = \dfrac{1}{\beta\underline{\lambda}} + \dfrac{I_1 - I_2}{r}$ and $\Gamma(\bar{\lambda}) = -\dfrac{1}{\beta\bar{\lambda}} + \dfrac{I_2}{r} < 0$ by the assumption of Theorem 6.3. From the equality in (12),

$$\begin{aligned} -\frac{1}{2}\theta^2\lambda^2\Gamma''(\lambda) - \lambda\Gamma'(\lambda)(\theta^2 + \beta + \delta - r) + r\Gamma(\lambda) + \frac{\delta}{\beta}\frac{\Gamma'(\lambda)}{\Gamma(\lambda) + \frac{1}{\beta\lambda}} \\ = -\frac{\delta}{\beta\lambda} - (I_1 - I_2) + \frac{\delta}{\beta^2}\frac{1}{\Gamma(\lambda) + \frac{1}{\beta\lambda}}\frac{1}{\lambda^2}. \end{aligned} \tag{42}$$

Because we take the assumption of (40), $G'(\lambda) < 0$, i.e., $G(\lambda)$ is monotonic decreasing, as a result,

$$\frac{1}{\Gamma(\lambda) + \frac{1}{\beta\lambda}} = \frac{1}{G(\lambda) - (I_1 - I_2)/r} \leq \frac{r}{I_2}.$$

Then the right-hand side of (42) satisfies the inequality

$$\begin{aligned} &-\frac{\delta}{\beta\lambda} - (I_1 - I_2) + \frac{\delta}{\beta^2}\frac{1}{\Gamma(\lambda) + \frac{1}{\beta\lambda}}\frac{1}{\lambda^2} \\ &\leq -\frac{\delta}{\beta\lambda}\Big(1 - \frac{r}{I_2}\frac{1}{\beta\lambda}\Big) - (I_1 - I_2) \\ &\leq -\frac{\delta}{\beta\lambda}\Big(1 - \frac{r}{I_2}\frac{1}{\beta\bar{\lambda}}\Big) - (I_1 - I_2) \\ &< -\frac{\delta}{\beta\lambda}\Big(1 - \frac{r}{I_2}\frac{I_2}{r}\Big) - (I_1 - I_2) < 0. \end{aligned}$$

Applying the comparison principle of Friedman (1982) to (42) we get $\Gamma(\lambda) < 0$ for $\underline{\lambda} < \lambda < \bar{\lambda}$, which is equivalent to (41).

Now we verify the first inequality in (10). The case in which $0 \leq x < \hat{x}$ is trivial. For $x \geq \hat{x}$, we obtain the equality

$$\begin{aligned} (\beta + \delta)\phi(x) - (rx + I_1)\phi'(x) + \frac{\theta^2}{2}\frac{\phi'(x)^2}{\phi''(x)} + 1 + \ln\phi'(x) - \delta U_2(x) \\ = -\frac{I_1 - I_2}{\beta(x + \frac{I_2}{r})} + \ln K, \end{aligned}$$

because $\phi(x) = U_2(x)$. The term $-\dfrac{I_1 - I_2}{\beta(x + \frac{I_2}{r})} + \ln K$ is monotonic increasing function of $x$, so the assumption $\underline{\lambda} \leq \ln K/(I_1 - I_2)$ gives

$$-\frac{I_1 - I_2}{\beta(x + \frac{I_2}{r})} + \ln K \geq 0.$$

**Q.E.D.**

### 6.8.4 Convergence of the Iterative Procedure

We show that the approximate function $G(\cdot)$ obtained from the iterative procedure converges to the implicit equation (13) by using the Banach fixed-point theorem.

Consider a domain $X = [\underline{\lambda}, \bar{\lambda}]$ of $\lambda(\cdot)$. We denote $\mathcal{R}$ by the set of real numbers and $\mathcal{B}(X, \mathcal{R})$ by the set of all bounded functions $g : X \to \mathcal{R}$. Because $\mathcal{R}$ is complete, so $\mathcal{B}(X, \mathcal{R})$ with the supremum norm

$$d(g, h) \equiv \sup\{|g(x) - h(x)| : x \in X\}$$

is a complete metric space. Let $\mathcal{C}(X, \mathcal{R})$ be the set of all continuous bounded functions $f : X \to \mathcal{R}$. Then $\mathcal{C}(X, \mathcal{R})$ is a closed subspace of $\mathcal{B}(X, \mathcal{R})$, so that $\mathcal{C}(X, \mathcal{R})$ is also a complete metric space. Thus, the continuous and decreasing function $G(\lambda)$, which is a solution to the differential equation (12) satisfying

$$\frac{I_1}{r} \leq G(\lambda) \leq \frac{1}{\beta\underline{\lambda}} + \frac{I_1 - I_2}{r}$$

should be in $\mathcal{C}(X, \mathcal{R})$.

Define for any $G(\lambda) \in \mathcal{C}(X, \mathcal{R})$

$$\begin{aligned} T\big(G(\lambda)\big) \equiv \quad & \frac{1}{\lambda(\beta + \delta)} + B(\underline{\lambda})\lambda^{-\alpha_\delta} + B^*(\bar{\lambda})\lambda^{-\alpha_\delta^*} \\ & + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\Big[(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_{\underline{\lambda}}^{\lambda} \mu^{\alpha_\delta - 2} \ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu \\ & + (\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\bar{\lambda}} \mu^{\alpha_\delta^* - 2} \ln\Big\{\beta\Big(G(\mu) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big\}d\mu\Big]. \end{aligned}$$

Then $T$ is continuous and is in $\mathcal{C}(X, \mathcal{R})$ from

$$|T\big(G(\lambda)\big)| \leq \frac{2\delta}{\theta^2\beta}\frac{(\bar{\lambda} - \underline{\lambda})}{\underline{\lambda}\,\bar{\lambda}} \sup_{\lambda} |T\big(G(\lambda)\big)|.$$

If we assume that

$$\frac{2\delta}{\theta^2\beta}\frac{(\bar{\lambda} - \underline{\lambda})}{\underline{\lambda}\,\bar{\lambda}} < 1,$$

then $T : \mathcal{C}(X, \mathcal{R}) \to \mathcal{C}(X, \mathcal{R})$ is a contraction mapping. This is because for any $G_1(\lambda),\ G_2(\lambda) \in \mathcal{C}(X, \mathcal{R})$, $T$ satisfies

$$\sup_{\lambda} |T\big(G_1(\lambda)\big) - T\big(G_2(\lambda)\big)| = \frac{2\delta}{\theta^2\beta} \frac{(\bar{\lambda} - \underline{\lambda})}{\underline{\lambda}\ \bar{\lambda}} \sup_{\lambda} |G_1(\lambda) - G_2(\lambda)|.$$

Let $G_i(\lambda)$ be a function and $B_i(\underline{\lambda}_i),\ B_i^*(\bar{\lambda}_i),\ \underline{\lambda}_i,\ \bar{\lambda}_i$ be the constants obtained from $i$-th iteration. We verify that $G_i(\lambda)$ converges uniformly to $G(\lambda)$ on $[\underline{\lambda}, \bar{\lambda}]$ by the Banach fixed-point theorem. Then clearly we have that $B_i(\underline{\lambda}_i) \to B(\underline{\lambda}),\ B_i^*(\bar{\lambda}_i) \to B^*(\bar{\lambda}),\ \underline{\lambda}_i \to \underline{\lambda}$, and $\bar{\lambda}_i \to \bar{\lambda}$ as $i \to \infty$.

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| $\delta \setminus x$ | $\hat{x}-45$ | $\hat{x}-40$ | $\hat{x}-35$ | $\hat{x}-30$ | $\hat{x}-25$ | $\hat{x}-20$ | $\hat{x}-15$ | $\hat{x}-10$ | $\hat{x}-5$ | $\hat{x}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.5918 | 0.7787 | 0.9490 | 1.1103 | 1.2657 | 1.4168 | 1.5646 | 1.7097 | 1.8526 | 1.9936 |
| 0.005 | 0.4946 | 0.6927 | 0.8681 | 1.0326 | 1.1903 | 1.3434 | 1.4927 | 1.6391 | 1.7831 | 1.9250 |
| 0.006 | 0.4922 | 0.6901 | 0.8654 | 1.0299 | 1.1878 | 1.3408 | 1.4902 | 1.6367 | 1.7807 | 1.9227 |
| 0.007 | 0.4899 | 0.6876 | 0.8628 | 1.0274 | 1.1853 | 1.3384 | 1.4879 | 1.6344 | 1.7785 | 1.9204 |
| 0 | 28.9932 | 43.4261 | 56.6174 | 69.2027 | 81.4303 | 93.4243 | 105.2557 | 116.9686 | 128.5927 | 140.1485 |
| 0.005 | 22.0918 | 37.2083 | 50.6002 | 63.2734 | 75.5535 | 87.5893 | 99.4613 | 111.2181 | 122.8909 | 134.5011 |
| 0.006 | 22.0244 | 37.0867 | 50.4454 | 63.0975 | 75.3645 | 87.3928 | 99.2619 | 111.0193 | 122.6957 | 134.3120 |
| 0.007 | 21.9572 | 36.9664 | 50.2930 | 62.9247 | 75.1791 | 87.2004 | 99.0669 | 110.8252 | 122.5055 | 134.1282 |

Table 1: **Optimal consumption (top table) and optimal risky investment (bottom table) as a function of initial wealth $x$ for several values of $\delta$.** The amounts of optimal consumption and risky investment decrease as the intensity $\delta$ of forced unemployment increases. Importantly, individuals could be significantly affected by risk of forced unemployment even if the possibility of unemployment is very small. Default parameter values: $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $\mu = 0.1123$ (expected rate of stork returns), $\sigma = 0.1954$ (stock volatility), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

| | $\mu$ | | | $\sigma$ | | | $K$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $\delta$ | 0.1023 | 0.1123 | 0.1223 | 0.1854 | 0.1954 | 0.2054 | 2 | 3 | 4 |
| 0 | 47.4590 | 51.1636 | 54.5313 | 52.5721 | 51.1636 | 49.8445 | 90.0721 | 51.1636 | 37.8977 |
| 0.005 | 46.0740 | 49.7891 | 53.1892 | 51.2084 | 49.7891 | 48.4632 | 87.1063 | 49.7891 | 37.0099 |
| 0.006 | 45.8201 | 49.5342 | 52.9377 | 50.9543 | 49.5342 | 48.2083 | 86.5632 | 49.5342 | 36.8440 |
| 0.007 | 45.5731 | 49.2850 | 52.6911 | 50.7057 | 49.2850 | 47.9592 | 86.0340 | 49.2850 | 36.6814 |

Table 2: **Critical wealth levels $\hat{x}$ for various parameter values of $\mu$, $\sigma$, $K$, and $\delta$.** The critical wealth level at which an individual chooses to retire increases as expected return rates increase or as stock volatility decreases (or both). Increase in preference for leisure after retirement may have the opposite effect. Default parameter values: $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

| | $\mu$ | | | $\sigma$ | | | K | | |
|---|---|---|---|---|---|---|---|---|---|
| Percentile of net worth | 0.1023 | 0.1123 | 0.1223 | 0.1854 | 0.1954 | 0.2054 | 2 | 3 | 4 |
| 0-25 | 0.7978 | 0.5942 | 0.4433 | 0.5279 | 0.5942 | 0.6617 | 0.6617 | 0.5942 | 0.4917 |
| 25-49.9 | 0.2792 | 0.2105 | 0.1590 | 0.1878 | 0.2105 | 0.2334 | 0.2754 | 0.2105 | 0.1721 |
| 50-74.9 | 0.1571 | 0.1201 | 0.0919 | 0.1077 | 0.1201 | 0.1325 | 0.1606 | 0.1201 | 0.0964 |
| 75-89.9 | 0.0933 | 0.0723 | 0.0560 | 0.0652 | 0.0723 | 0.0794 | 0.1000 | 0.0723 | 0.0565 |
| 90-100 | 0.0606 | 0.0476 | 0.0374 | 0.0431 | 0.0476 | 0.0520 | 0.0687 | 0.0476 | 0.0359 |

Table 3: **Reservation purchase price (RPP) to wealth ratio, $\epsilon(x)/x$, for various parameter values of $\mu$, $\sigma$, and $K$.** An individual's RPP increases as her net worth decreases; this result implies that a person with low net worth will pay much more to obtain insurance coverage than will a person with high net worth. Default parameter values: $\delta = 0.005$ (unemployment intensity), $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

| | $\mu$ | | | $\sigma$ | | | K | | |
|---|---|---|---|---|---|---|---|---|---|
| Percentile of net worth | 0.1023 | 0.1123 | 0.1223 | 0.1854 | 0.1954 | 0.2054 | 2 | 3 | 4 |
| 0-25 | 0.2489 | 0.1710 | 0.1171 | 0.1469 | 0.1710 | 0.1965 | 0.1965 | 0.1710 | 0.1374 |
| 25-49.9 | 0.0849 | 0.0592 | 0.0410 | 0.0510 | 0.0592 | 0.0676 | 0.0800 | 0.0592 | 0.0469 |
| 50-74.9 | 0.0461 | 0.0326 | 0.0230 | 0.0283 | 0.0326 | 0.0371 | 0.0451 | 0.0326 | 0.0254 |
| 75-89.9 | 0.0261 | 0.0188 | 0.0134 | 0.0164 | 0.0188 | 0.0212 | 0.0268 | 0.0188 | 0.0142 |
| 90-100 | 0.0162 | 0.0118 | 0.0086 | 0.0104 | 0.0118 | 0.0133 | 0.0175 | 0.0118 | 0.0087 |

Table 4: **Individual welfare benefit (IWB) of market innovation to wealth ratio, $\nabla(x)/x$, for various parameter values of $\mu$, $\sigma$, and $K$.** The IWB decreases with an individual's net worth; this trend implies that the introduction of private unemployment insurance might increase drastically the welfare of the poor. Default parameter values: $\delta = 0.005$ (unemployment intensity), $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

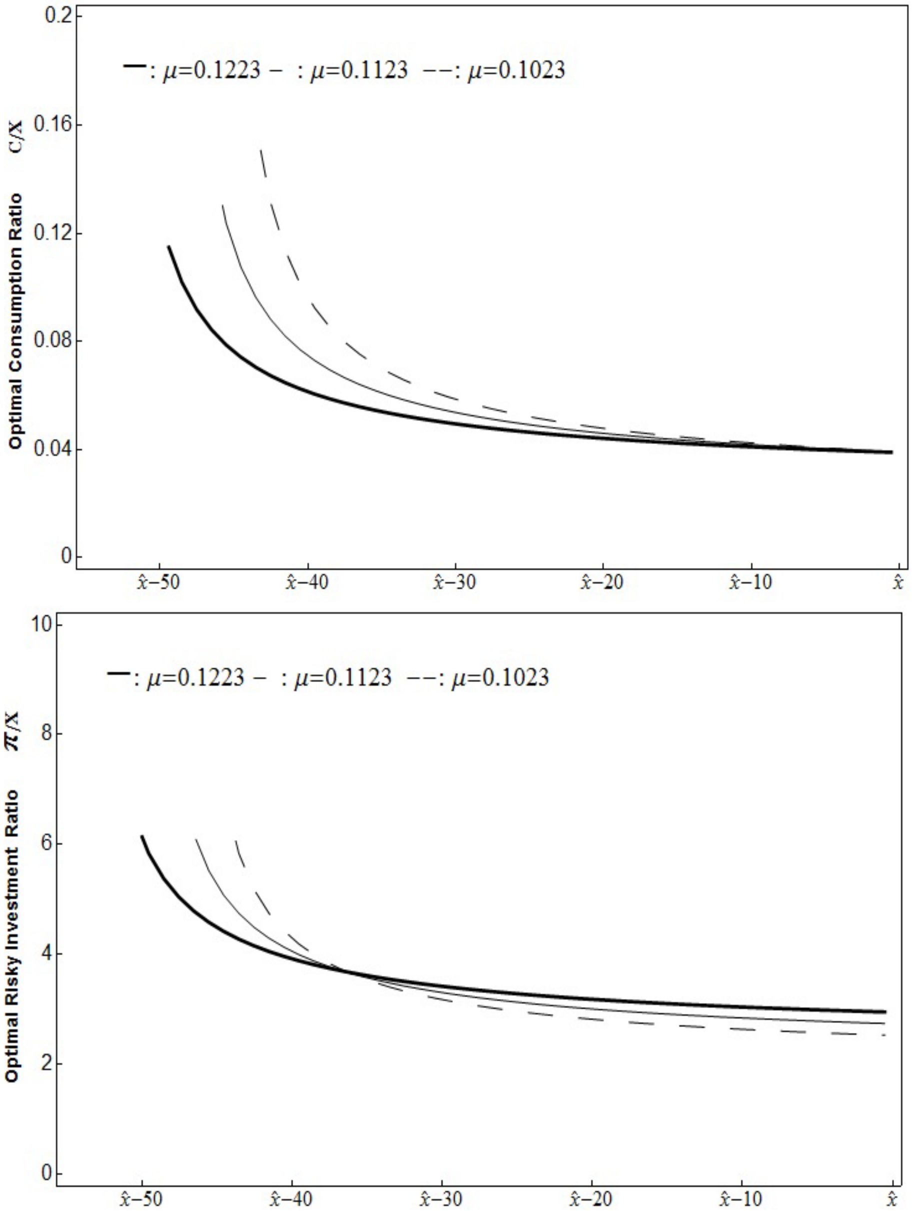


Figure 1: **Optimal consumption to wealth ratio and risky investment to wealth ratio as a function of initial wealth $x$ for various expected rates $\mu$ of stock returns**. The amounts of optimal consumption and risky investment are expected to increase as an individual's wealth increases. Default parameter values: $\delta = 0.005$ (unemployment intensity), $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $\sigma = 0.1954$ (stock volatility), $K = 3$ (post-retirement leisure preference), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

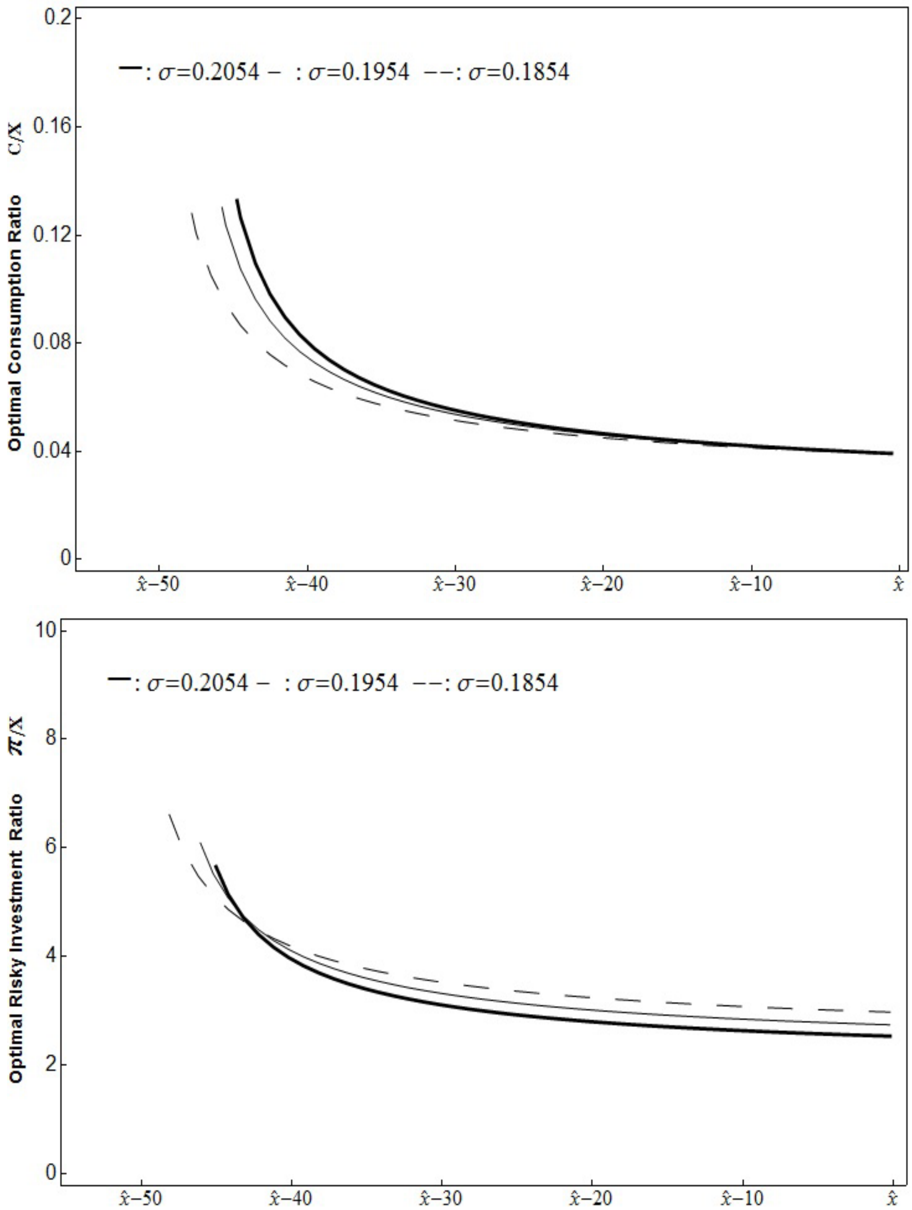


Figure 2: **Optimal consumption to wealth ratio and risky investment to wealth ratio as a function of initial wealth $x$ for various stock volatilities $\sigma$.** Consumption and investment ratios decrease more rapidly as the stock volatility increases. Default parameter values: $\delta = 0.005$ (unemployment intensity), $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $\mu = 0.1123$ (expected rate of stock returns), $K = 3$ (post-retirement leisure preference), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

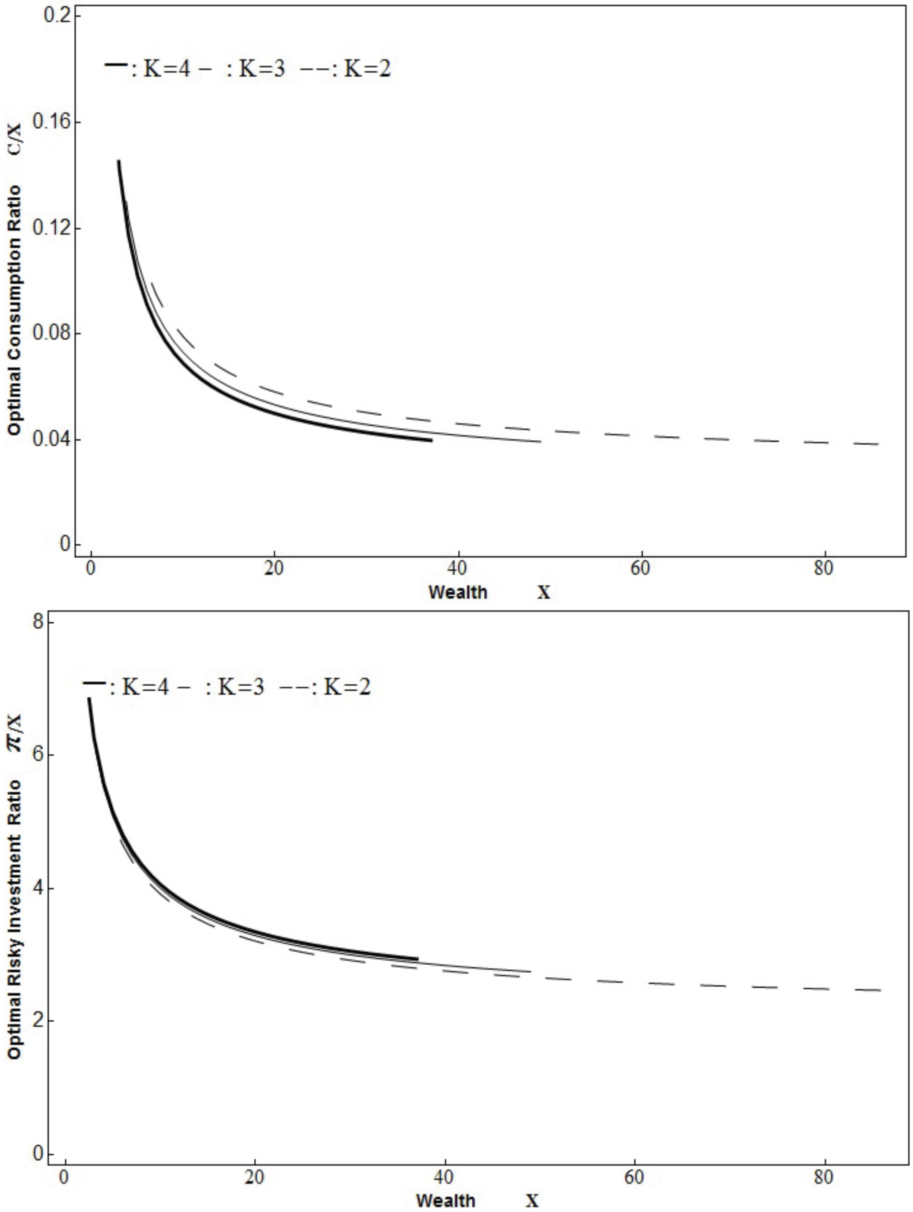


Figure 3: **Optimal consumption to wealth ratio and risky investment to wealth ratio as a function of initial wealth $x$ for various post-retirement leisure preferences $K$.** Consumption decreases and investment in the risky asset increases as the individual's leisure demand increases. Default parameter values: $\delta = 0.005$ (unemployment intensity), $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $\mu = 0.1123$ (expected rate of stock returns), $\sigma = 0.1954$ (stock volatility), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

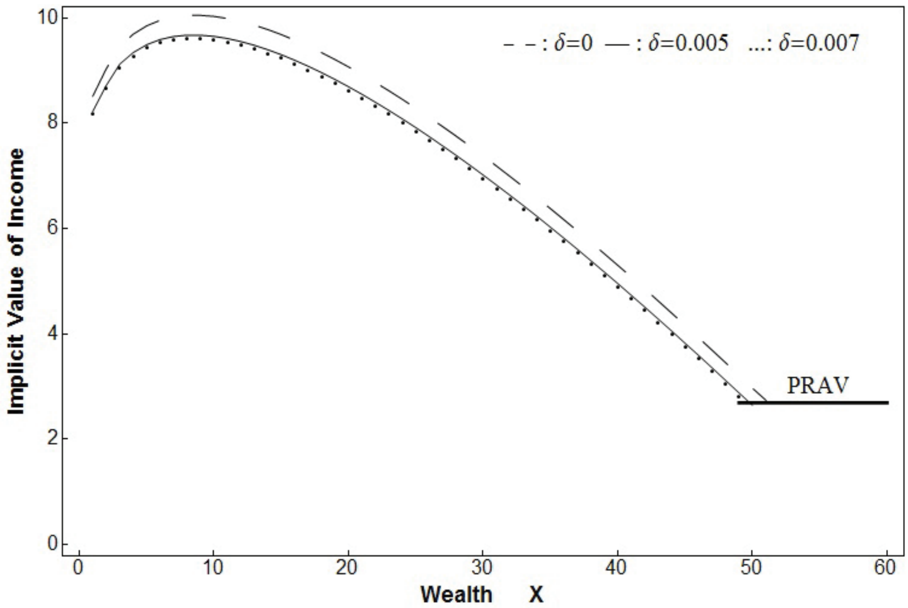


Figure 4: **Implicit value of income as a function of initial wealth** $x$. The implicit value of income increases as wealth increases up to a somewhat small level, then starts to decrease. Default parameter values: $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $\mu = 0.1123$ (expected rate of stock returns), $\sigma = 0.1954$ (stock volatility), $K = 3$ (post-retirement leisure preference), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

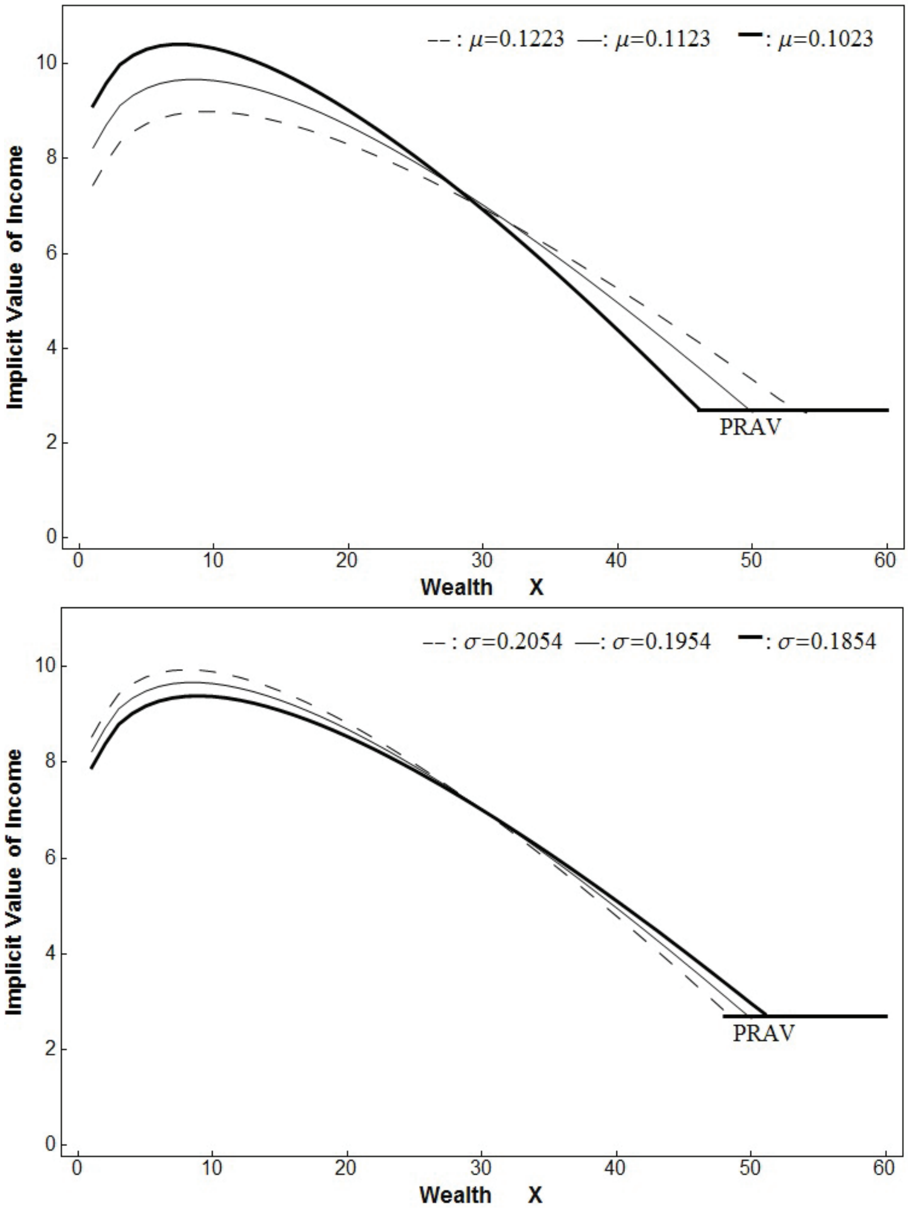


Figure 5: **Implicit value of income as a function of initial wealth $x$ for various parameter values of investment opportunity set ($\mu$ and $\sigma$).** The relationship between implicit value of income and investment opportunity reveals that the poor might prefer a high risk premium from the stock investment over uninsurable or unhedgeable labor income, whereas the opposite might be true for the rich who are about to retire. Default parameter values: $\delta = 0.005$ (unemployment intensity), $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $K = 3$ (post-retirement leisure preference), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

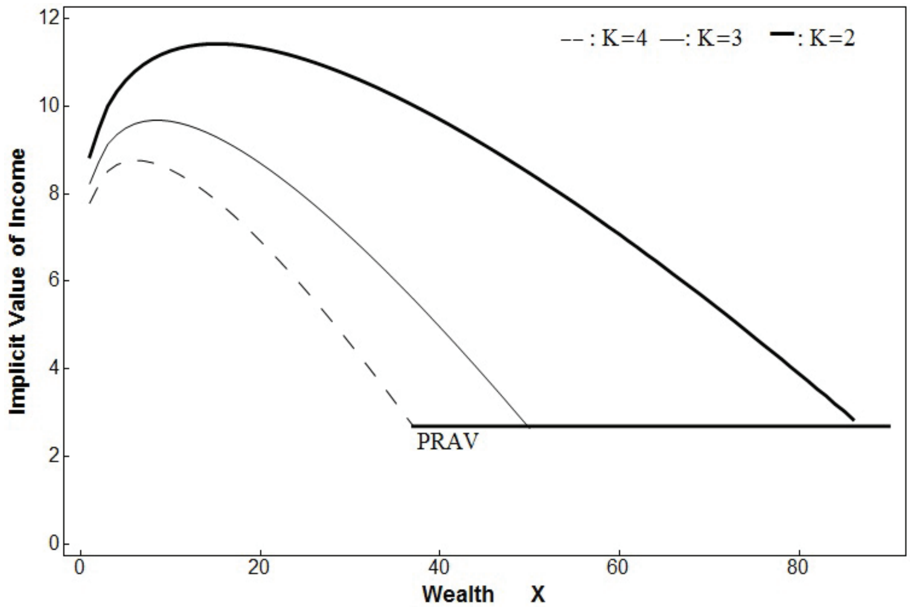


Figure 6: **Implicit value of income as a function of initial wealth $x$ for various values of post-retirement leisure preferences $K$.** As an individual's post-retirement leisure preference decreases, her willingness to delay retirement to increase utility gains from the labor income increases. Default parameter values: $\delta = 0.005$ (unemployment intensity) $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $\mu = 0.1123$ (expected rate of stock returns), $\sigma = 0.1954$ (stock volatility), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

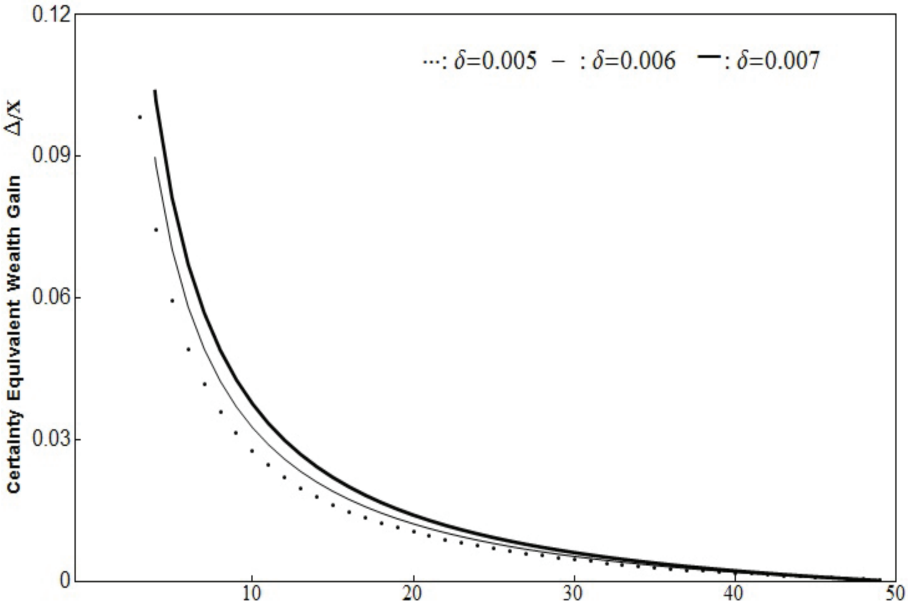


Figure 7: **Certainty equivalent wealth gain (CEWG) to wealth ratio, $\Delta(x)/x$, for various values of unemployment intensities** $\delta$. CEWG increases as $\delta$ increases. Default parameter values: $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $\mu = 0.1123$ (expected rate of stock returns), $\sigma = 0.1954$ (stock volatility), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

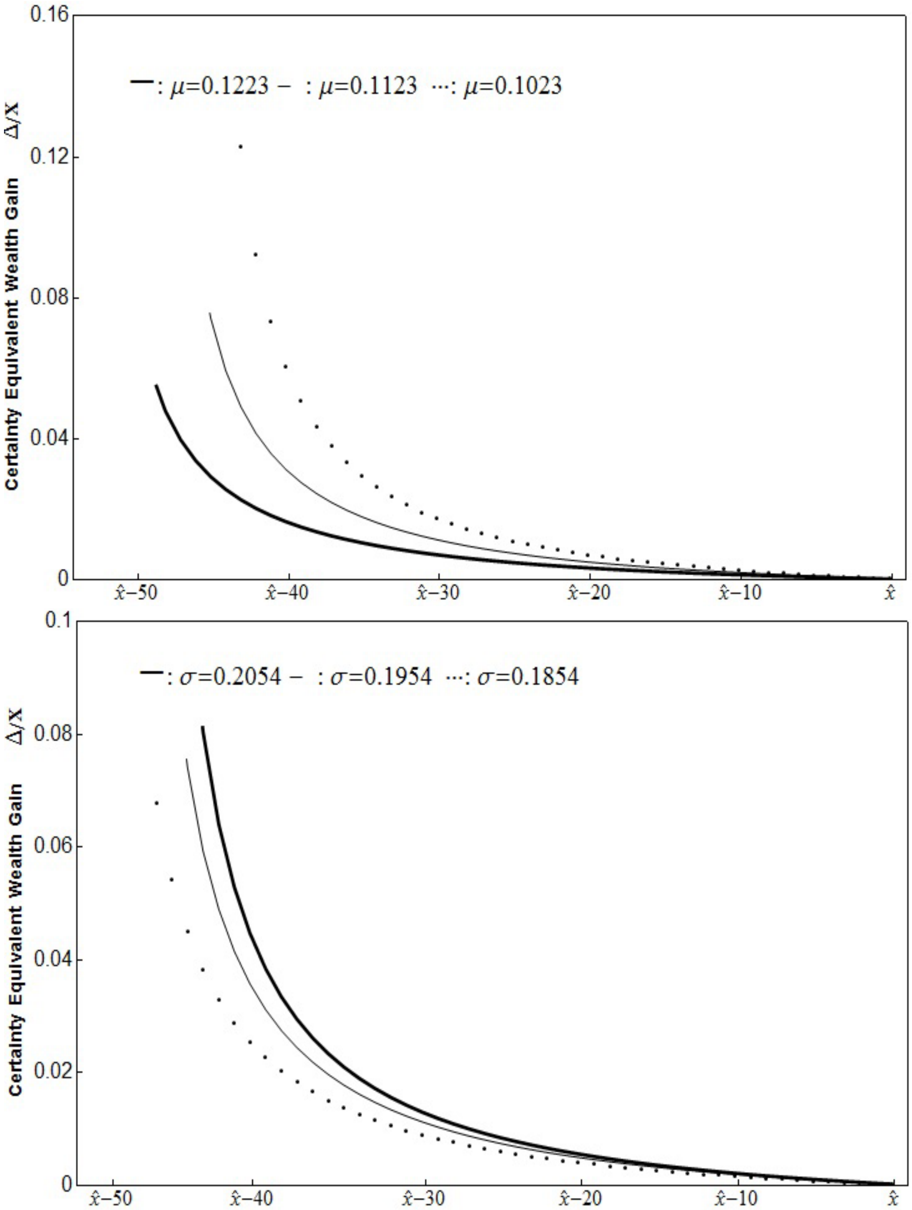


Figure 8: **Certainty equivalent wealth gain (CEWG) to wealth ratio, $\Delta(x)/x$, for various values of investment opportunity set ($\mu$ and $\sigma$).** CEWG increases as expected rate $\mu$ of stock returns decreases or stock volatility $\sigma$ increases (or both). Default parameter values: $\delta = 0.005$ (unemployment intensity), $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $K = 3$ (post-retirement leisure preference), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

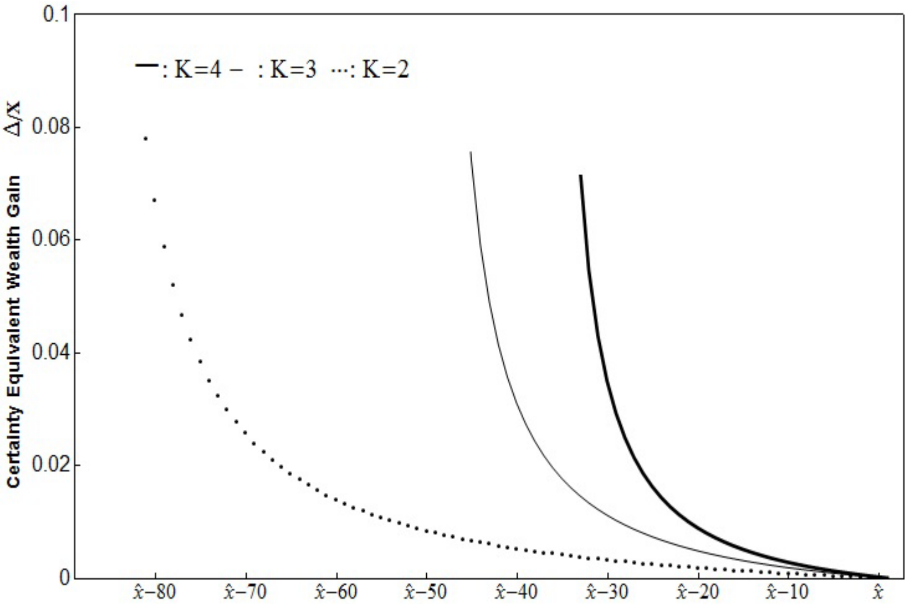


Figure 9: **Certainty equivalent wealth gain (CEWG) to wealth ratio, $\Delta(x)/x$, for various post-retirement leisure preferences** $K$. CEWG is sensitive to changes of $K$. As an individual's leisure demand $K$ increases, the CEWG required to buffer against unemployment risks increases. Default parameter values: $\delta = 0.005$ (unemployment intensity), $\beta = 0.0371$ (subjective discount rate), $r = 0.0371$ (risk-free interest rate), $\mu = 0.1123$ (expected rate of stock returns), $\sigma = 0.1954$ (stock volatility), $I_1 = 1$ (labor income), and $I_2 = 0.10$ (post-retirement income).

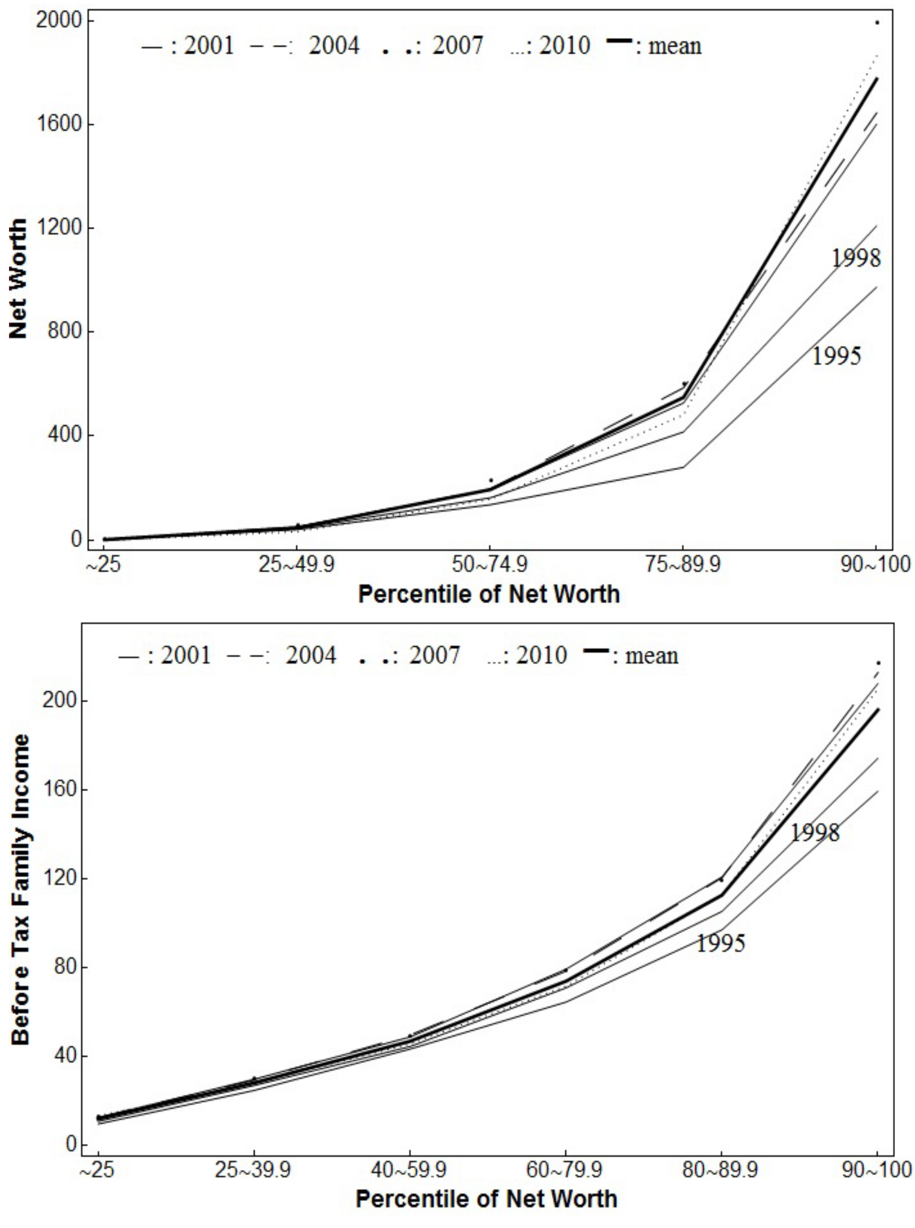


Figure 10: **Family net worth and before-tax family income by percentile of net worth from the SCF for the period 1995-2010.** Both family net worth and before-tax family income increase with percentile of net worth.