Realized Higher-Order Comoments

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Abstract

This paper provides estimators of the realized third and fourth order (joint) cumulants, which are standardized (co)moments, for arithmetic returns with one assumption under which each price is a martingale. The estimators that are developed based on Aggregation Property of Neuberger (2012) help to access the ex-post moments of returns for a specific period and do not require data for a long period. Moreover, we show that neither realized fourth moments nor third comoments of log returns exist under the similar condition. In addition, we conduct an empirical study based on the realized higher order cumulants and the results are consistent with the literature.

JEL Classifications: G11, G12, G13

Keywords: Realized Joint Cumulant, Realized Coskewness, Realized Kurtosis, Realized Cokurtosis, Aggregation Property.

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I. Introduction

Large size of sample can reduce estimation errors of sample moments. Therefore, if returns of assets are i.i.d., we can get accurate estimates of moments of returns by increasing the sample size. However, distribution of asset returns appears to be time varying and it is prominent when there is a shock in the market as a series of financial crises show (Engle 1982; Ang and Timmermann 2012; Baur 2012). Thus there is a tradeoff between sample size and outdated data.

In the case of the second moment, there are some remedies for this problem. One of them is putting more weight on the recent data as EWMA of J.P. Morgan or GARCH of Bollerslev (1986). Another solution is considering the recent high frequency data as Andersen et al. (2003) propose. When we estimate a variance of asset return for a specific period with returns in the period only, information of the other period is excluded. Therefore, it helps to understand the period clearly.[1] In addition, it helps to get moments of newly issued securities, which have limited data period. Basically, estimation of variance with high-frequency data is in line with the following equality for a martingale process *F*.

$$E_0\left[(F_T - F_0)^2\right] = E_0\left[\sum_{i=1}^{N}(F_{t_i} - F_{t_{i-1}})^2\right], 0 = t_0 < t_1 < \cdots < t_N = T \tag{1}$$

After Markowitz, second moments of returns have been used to measure risk of assets. However, many theoretical and empirical studies show that the third and the fourth moments are also related to returns of securities (Kraus and Litzenberger 1976; Harvey and Siddique 2000; Dittmar 2002; Ang et al. 2006; Conrad, Dittmar and Ghysels 2013). When a return of a securities follows normal distribution, the first and the second moments of the return are enough to understand distribution of the return fully. However, as many studies including Fama (1963) provide counterevidence of the normality, estimation of the third and the fourth moments has been an important issue.

Conrad, Dittmar and Ghysels (2013) obtain individual securities' implied skewness and kurtosis via method of Bakshi, Kapadia and Madan (2003) and they show a negative (positive) relation between skewness (kurtosis) and the subsequent return. Amaya et al. (2015) show the

[1] Some studies use realized variance to understand the characteristics of the market. For example, Jiang and Tian (2005) test whether the implied volatility really forecasts the realized volatility of the future period, and Bollerslev, Gibson and Zhou (2011) estimate volatility risk premium from the implied volatility and realized volatility.

similar result with the past realized skewness and kurtosis from high frequency data. However, aside from the interesting result, time horizon of the measures of Amaya et al. (2015) is arguable because their skewness is a sample skewness of a return for a short period, 5-minute, although coverage of the samples is equal to the time horizon of their study, a week.[2]

This problem arises because Equation (1) is not directly extended to the higher order moments. Therefore, in general, we require additional assumptions to get realized higher order moments. An example for the assumptions is that returns are i.i.d.. Then, realized third moment can be estimated with $\sum_{i=1}^{N} r_{t_i}^3$. However, Neuberger (2012) provides a counter evidence about it; according to the empirical result, third moment of *N*-period return does not appear to be *N* times of third moment of 1-period return. Hence when there is no undisputed assumption, high frequency data cannot yield appropriate moments of low frequency. For example, when we want to estimate the third moment of a return for a specific year, with daily returns of the year, it just provides one observation, a cube of a return for the year.

In the case of the third moment, this problem is relieved by Neuberger (2012). The author generalize Equation (1) with a name of Aggregation Property, and confirms that we cannot obtain moments over the second order when the provided information is limited to the price process *F*. Moreover, he shows that we can also obtain the third moment and no higher order moments when the information is extended to include variance process additionally. Inclusion of the variance process (only) is reasonable because of its importance as we see some derivatives are quoted with volatility of their underlying assets. However, implied higher order moments can also be obtained as Bakshi, Kapadia and Madan (2003) show. Furthermore, Conrad, Dittmar and Ghysels (2013) show the importance of the implied fourth moment as well as the third moment. As the finance literature emphasizes the importance of the fourth moment, we want to extend the study to get realized fourth moments.

Beside of the moments of individual assets, Harvey and Siddique (2000) emphasize coskewness as well as covariance. In the case of covariance, like Equation (1), we have the

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[2] From each 5-minute return r_{t_i}, their skewness and kurtosis are represented as $\left(\frac{1}{n}\sum_{i=1}^{N} r_{t_i}^k\right) \Big/ \left(\frac{1}{n}\sum_{i=1}^{N} r_{t_i}^2\right)^{k/2}$ with k=3 or 4 as they define realized k^{th} moment with $N^{k/2-1}\sum_{i=1}^{N} r_{t_i}^k$.

following relation given two martingale processes $F_{1,t}$ and $F_{2,t}$.

$$E_0\left[(F_{1,T}-F_{2,0})(F_{2,T}-F_{2,0})\right]=E_0\left[\sum_{i=1}^{N}(F_{1,t_i}-F_{1,t_{i-1}})(F_{2,t_i}-F_{2,t_{i-1}})\right], 0=t_0<\cdots<t_N=T \quad (2)$$

However, as above, higher order comoments are not obtainable just with price processes, $F_{1,t}$ and $F_{2,t}$. Neuberger (2011) investigates coskewness although it is omitted in the published version. The study provides new perspective with a new definition of coskewness as a sensitivity of expected realized skewness with respect to the investments. A point that it is defined without covariance process improves its accessibility and usefulness in the future studies. However, another point that the coskewness is not developed based on the traditional definition arises a weak link from the other studies so far. Therefore, we also investigate the realized joint moments, likewise coskewness or cokurtosis, in line with the literature.

This paper also presents empirical results. As mentioned above, BKM's methodology makes it possible to get higher order implied moments for a single security. Accordingly we can get realized higher order moments for a single security. By contrast, estimation of comoment has a practical problem. That is, while we require lower order implied comoments to get a realized comoment, implied comoments are hardly accessible because they require exotic options like basket options or spread options. We partially solve this problem by adopting Kempf, Korn and Saßning (2015).[3]

Finance literature mostly uses log returns instead of arithmetic returns because short term returns are easily transferred to long term returns and vice versa because of additivity of log returns. However, we do not require the transformation when the sample period coincides to the time horizon as we get the realized moments. In addition, arithmetic returns are more adequate than log returns when we deal with the asset allocation for a fixed time horizon. Therefore this paper concentrates on the moments of arithmetic returns although we also address the realized moments of log returns.

The rest of the paper is organized as follows. Section II reviews about the Aggregation Property and investigates some properties about higher-order moments and comoments. Section III discusses method to get implied moments or comoments in advance for estimation

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[3] Kempf, Korn and Saßning (2015) calculate implied covariance just with index and individual options through an assumption of index model and an additional assumption about idiosyncratic risk. As a result, they show that this covariance is effective in asset allocation.

of realized moments. Section IV presents empirical results about realized moments. Section V concludes this study.

II. The Aggregation Property Given Comoment Processes

Let $X = (X_t, 0 \le t \le T)$ be an adapted vector valued stochastic process defined on a filtration. Then, the Aggregation Property is defined as follows.

Definition 1. The Aggregation Property (Neuberger 2012)

A function *g* on a vector valued process *X* has the *Aggregation Property* if and only if

$$E_s[g(X_u - X_s)] = E_s[g(X_u - X_t)] + E_s[g(X_t - X_s)], \ \forall(s,t,u) \text{ s.t. } 0 \le s \le t \le u \le T. \quad (3)$$

When the definition is combined with a law of iteration, we have a following relation for any partition $0 = t_0 \le t_1 \le \cdots \le t_N = T$:

$$E_0[g(X_T - X_0)] = E_0\left[\sum_{j=1}^{N} g(\Delta X_j)\right].^{4} \quad (4)$$

Therefore

$$\sum_{j=1}^{N} g(\Delta X_j) \quad (5)$$

is called a realized measure of

$$E_0[g(X_T - X_0)] \quad (6)$$

because the expression (5) is an ex-post estimator of (6). For example, for a martingale process *S*,

[4] Hereafter, for a convenience, t_j is denoted by j and $X_j - X_{j-1}$ and $X_{\cdot,j} - X_{\cdot,j-1}$ are denoted by ΔX_j and $\Delta X_{\cdot,j}$ respectively.

$$\sum_{j=1}^{N}(\Delta S_j)^2 \tag{7}$$

is an unbiased estimator of

$$E_0\left[(S_T - S_0)^2\right] \tag{8}$$

because $g(x) = x^2$ satisfies Equation (3). Similarly, when $V_t = \mathrm{var}_t(S_T)$,

$$\sum_{j=1}^{N}\left((\Delta S_j)^3 + 3\Delta S_j \Delta V_j\right) \tag{9}$$

is an estimator of

$$E_0\left[(S_T - S_0)^3\right] \tag{10}$$

because $g(\Delta S, \Delta V) = (\Delta S)^3 + 3\Delta S \Delta V$ satisfies Equation (3) and the following holds:

$$\begin{aligned} E_0\left[(S_T - S_0)^3\right] &= E_0\left[(S_T - S_0)^3 + 3(V_T - V_0)(S_T - S_0)\right] \\ &= E_0\left[\sum_{j=1}^{N}\left((\Delta S_j)^3 + 3\Delta V_j \Delta S_j\right)\right] \end{aligned} . \tag{11}$$

Therefore Neuberger names (9) as realized third moment, (10) as a true third moment, and $E_0^Q\left[(S_T - S_0)^3\right]$, which is obtained from prices of derivatives, as an implied third moment.

Neuberger shows that there are no additional higher order moments when the information process X_t is defined as (S_t, V_t). It implies that we may get realized higher moments or comoments if we extend the information process X_t. Because of the importance of the third and fourth (co)moments, we investigate the (co)moments given information of lower order moments. Therefore let X_t be a vector valued process $\{(S_{1,t}, S_{2,t}, \vec{M}_t')' : 0 \le t \le T\}$, where

$$\vec{M} = (M_{2,0}, M_{1,1}, M_{0,2}, M_{3,0}, M_{2,1}, M_{1,2}, M_{0,3})' \tag{12}$$

with

$$M_{k,l}(t) = E_t\left[(S_1(T) - S_1(t))^k (S_2(T) - S_2(t))^l\right]. \tag{13}$$

Then, Proposition 1 provides a general form of functions which have the Aggregation Property on X.

Proposition 1. When S_1 and S_2 are martingale processes, a two dimensional analytic function *g* has the Aggregation Property on the vector valued process *X* if and only if *g* can be represented as follows:

$$
\begin{aligned}
g\left(\Delta S_1, \Delta S_2, \Delta M\right) = {} & h_1 \Delta S_1 + h_2 \Delta S_2 + h_3 (\Delta S_1)^2 + h_4 \Delta M_{2,0} + h_5 \Delta S_1 \Delta S_2 \\
& + h_6 \Delta M_{1,1} + h_7 (\Delta S_2)^2 + h_8 \Delta M_{0,2} + h_9 ((\Delta S_1)^3 + 3 \Delta S_1 \Delta M_{2,0}) \\
& + h_{10} \Delta M_{3,0} + h_{11} ((\Delta S_1)^2 \Delta S_2 + 2 \Delta S_1 \Delta M_{1,1} + \Delta S_2 \Delta M_{2,0}) + h_{12} \Delta M_{2,1} \\
& + h_{13} (\Delta S_1 (\Delta S_2)^2 + 2 \Delta S_2 \Delta M_{1,1} + \Delta S_1 \Delta M_{0,2}) + h_{14} \Delta M_{1,2} \\
& + h_{15} ((\Delta S_2)^3 + 3 \Delta S_2 \Delta M_{0,2}) + h_{16} \Delta M_{0,3} \\
& + h_{17} ((\Delta S_1)^4 + 6 (\Delta S_1)^2 \Delta M_{2,0} + 4 \Delta S_1 \Delta M_{3,0} + 3 (\Delta M_{2,0})^2) \\
& + h_{18} \begin{pmatrix} (\Delta S_1)^3 \Delta S_2 + \Delta S_2 \Delta M_{3,0} + 3 (\Delta M_{1,1} (\Delta S_1)^2 \\ + \Delta M_{2,1} \Delta S_1 + \Delta M_{2,0} \Delta S_1 \Delta S_2 + \Delta M_{2,0} \Delta M_{1,1}) \end{pmatrix} \\
& + h_{19} \begin{pmatrix} ((\Delta S_1)^2 + \Delta M_{2,0}) ((\Delta S_2)^2 + \Delta M_{0,2}) + 2 (\Delta M_{1,1})^2 \\ + 4 \Delta M_{1,1} \Delta S_1 \Delta S_2 + 2 \Delta M_{1,2} \Delta S_1 + 2 \Delta M_{2,1} \Delta S_2 \end{pmatrix} \\
& + h_{20} \begin{pmatrix} \Delta S_1 (\Delta S_2)^3 + \Delta S_1 \Delta M_{0,3} + 3 (\Delta M_{1,1} (\Delta S_2)^2 \\ + \Delta M_{1,2} \Delta S_2 + \Delta M_{0,2} \Delta S_1 \Delta S_2 + \Delta M_{0,2} \Delta M_{1,1}) \end{pmatrix} \\
& + h_{21} ((\Delta S_2)^4 + 6 (\Delta S_2)^2 \Delta M_{0,2} + 4 \Delta S_2 \Delta M_{0,3} + 3 (\Delta M_{0,2})^2)
\end{aligned} \tag{14}
$$

for some constants $h_i, i \in \{1,...,21\}$.

Proof is in the Appendix.

For convenience, when we arrange the terms which is related to the $S_1^k S_2^l$ with $k \geq l$, the function with the Aggregation Property is represented with 12 individual terms as follows:

$$
\begin{aligned}
g(\Delta S_1, \Delta S_2, \Delta M) = {} & h_1 \Delta S_1 + h_2 (\Delta S_1)^2 + h_3 \Delta M_{2,0} + h_4 \Delta S_1 \Delta S_2 + h_5 \Delta M_{1,1} \\
& + h_6 ((\Delta S_1)^3 + 3\Delta S_1 \Delta M_{2,0}) + h_7 \Delta M_{3,0} \\
& + h_8 ((\Delta S_1)^2 \Delta S_2 + 2\Delta S_1 \Delta M_{1,1} + \Delta S_2 \Delta M_{2,0}) + h_9 \Delta M_{2,1} \\
& + h_{10} ((\Delta S_1)^4 + 6(\Delta S_1)^2 \Delta M_{2,0} + 4\Delta S_1 \Delta M_{3,0} + 3(\Delta M_{2,0})^2) \\
& + h_{11} \begin{pmatrix} (\Delta S_1)^3 \Delta S_2 + \Delta S_2 \Delta M_{3,0} + 3(\Delta M_{1,1} (\Delta S_1)^2 \\ + \Delta M_{2,1} \Delta S_1 + \Delta M_{2,0} \Delta S_1 \Delta S_2 + \Delta M_{2,0} \Delta M_{1,1}) \end{pmatrix} \\
& + h_{12} \begin{pmatrix} ((\Delta S_1)^2 + \Delta M_{2,0})((\Delta S_2)^2 + \Delta M_{0,2}) + 2(\Delta M_{1,1})^2 \\ + 4\Delta M_{1,1} \Delta S_1 \Delta S_2 + 2\Delta M_{1,2} \Delta S_1 + 2\Delta M_{2,1} \Delta S_2 \end{pmatrix}
\end{aligned}
\tag{14'}
$$

In Equation (14’), the Aggregation Property of ΔS_1, $(\Delta S_1)^2$ and $(\Delta S_1)^3 + 3\Delta S_1 \Delta M_{2,0}$ are shown in the Neuberger, and the Aggregation Property of $\Delta M_{k,l}$ is obvious. Among remainder terms, $\Delta S_1 \Delta S_2$ is a well-known estimator of covariance. The 8th, 10th, 11th, and 12th terms are introduced from this paper, to the best of our knowledge.

First, let us deal with the 8th term. We can observe that $(\Delta S_1)^2 \Delta S_2$ does not solely appear and requires additional terms. Furthermore, when a function g is defined as

$$
g(\Delta X) = (\Delta S_1)^2 \Delta S_2 + 2\Delta S_1 \Delta M_{1,1} + \Delta S_2 \Delta M_{2,0}, \tag{15}
$$

the following equation holds

$$
E_0[g(X(S_{1,T}, S_{2,T}) - X(S_{1,0}, S_{2,0}))] = E_0[(S_{1,T} - S_{1,0})^2 (S_{2,T} - S_{2,0})]. \tag{16}
$$

Therefore, although $\sum_{j=1}^{N} (\Delta S_{1,j})^2 \Delta S_{2,j}$ is related to the third comoment, it is not unbiased toward the true realized third comoment. To be an unbiased estimator, additional terms should be added. In other words, the following form plays as a realized third comoment

$$
cTM_{1,1,2}^{real} \equiv \sum_{j=1}^{N} \left((\Delta S_{1,j})^2 \Delta S_{2,j} + \Delta S_{2,j} \Delta M_{2,0,j} + 2\Delta S_{1,j} \Delta M_{1,1,j} \right). \tag{17}
$$

where $cTM_{a,b,c}$ denotes an estimator of

$$
E_0[(S_{a,T} - S_{a,0})(S_{b,T} - S_{b,0})(S_{c,T} - S_{c,0})].^5 \tag{18}
$$

5 The superscript ‘real’ represents that it is a realized moment. Afterward, *TM* is replaced by *FM* for the fourth cumulant which is linked to the fourth moment. Accordingly, FM_a denotes an estimator of the fourth cumulant

Therefore, if S_1 becomes more (less) volatile while S_2 is increasing, third comoment during [0, T] is higher (lower). In addition, if the covariance between returns is increasing (decreasing) while S_2 is increasing, third comoment during [0, T] is higher (lower). From a series of financial crises, we observe that covariance of returns is increasing when one of the return decreases, which is known as contagion effect and interdependence (Allen and Gale 2000; Forbes and Rigobon 2002; Cespa and Foucault 2014). It implies that if we measure third comoment with $\sum_{j=1}^{N}\left(\Delta S_{1,j}\right)^2 \Delta S_{2,j}$ rather than the expression (17), it could cause a positive bias.

Now let us investigate the fourth moment and comoments. Because $M_{k,l,T}=0$, we have a following relation from the 10th term of Equation (14').

$$E_0\begin{bmatrix}(S_{1,T}-S_{1,0})^4+6(S_{1,T}-S_{1,0})^2(M_{2,0,T}-M_{2,0,0}) \\ +4(S_{1,T}-S_{1,0})(M_{3,0,T}-M_{3,0,0})+3(M_{2,0,T}-M_{2,0,0})^2\end{bmatrix}$$
$$=E_0\left[(S_{1,T}-S_{1,0})^4\right]-3\left(E_0[(S_{1,T}-S_{1,0})^2]\right)^2 \tag{19}$$

Light hand side of Equation (19) is not the fourth moment. However, it is also an important measure, the fourth cumulant, which is a numerator of kurtosis. Therefore

$$FM_1^{real}\equiv\sum_{j=1}^{N}\left((\Delta S_{1,j})^4+6(\Delta S_{1,j})^2\Delta M_{2,0,j}+4\Delta S_{1,j}\Delta M_{3,0,j}+3(\Delta M_{2,0,j})^2\right) \tag{20}$$

is an estimator of the fourth cumulant, realized fourth cumulant. Like the third moment case, $(\Delta S_{1,j})^4$ does not solely appear in the fourth cumulant. Among the other terms, $6(\Delta S_{1,j})^2\Delta M_{2,0,j}+3(\Delta M_{2,0,j})^2$ is related to the adjustment about cumulant because

$$E_0\left[6(S_{1,T}-S_{1,0})^2(M_{2,0,T}-M_{2,0,0})+3(M_{2,0,T}-M_{2,0,0})^2\right]=-3M_{2,0,0}^2. \tag{21}$$

In addition, from the relation, $M_{2,0,j-1}=E_j[M_{2,0,j}+(\Delta S_{1,j})^2]$, if we allow the following approximation:

$$6(\Delta S_{1,j})^2\Delta M_{2,0,j}+3(\Delta M_{2,0,j})^2\approx 3\operatorname{cov}(\Delta M_{2,0,j},(\Delta S_{1,j})^2), \tag{22}$$

Therefore the fourth cumulant is greater when variance (until the maturity) and instantaneous

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of $S_{a,T}$ - $S_{a,0}$ and $cFM_{a,b,c,d}$ denotes an estimator which is related to the fourth comoment, $E_0[(S_{a,T}-S_{a,0})(S_{b,T}-S_{b,0})(S_{c,T}-S_{c,0})(S_{d,T}-S_{d,0})]$

variance are more correlated. Accordingly, omitting this term, or replacing this term with variance square, can cause a negative bias about the fourth cumulant because empirical studies report positive autocorrelation of variance, which is known as volatility clustering of Mandelbrot (1963). The remainder in Equation (20), $4\Delta S_{1,j}\Delta M_{3,0,j}$, implies that fourth cumulant is greater when return and change of the third moment are more correlated.

In the same manner, realized fourth order joint cumulants are defined from the 11th and 12th term of Equation (14') as follows:[6]

$$\begin{aligned} cFM^{real}_{1,1,1,2} \equiv & \sum_{j=1}^{N}\left((\Delta S_{1,j})^3\Delta S_{2,j} + \Delta S_{2,j}\Delta M_{3,0,j}\right) \\ & + 3\sum_{j=1}^{N}(\Delta M_{1,1,j}(\Delta S_{1,j})^2 + \Delta M_{2,1,j}\Delta S_{1,j} + \Delta M_{2,0,j}\Delta S_{1,j}\Delta S_{2,j} + \Delta M_{2,0,j}\Delta M_{1,1,j}) \end{aligned} \tag{23}$$

and

$$\begin{aligned} cFM^{real}_{1,1,2,2} \equiv & \sum_{j=1}^{N}((\Delta S_{1,j})^2 + \Delta M_{2,0,j})((\Delta S_{2,j})^2 + \Delta M_{0,2,j}) \\ & + \sum_{j=1}^{N}(2(\Delta M_{1,1,j})^2 + 4\Delta M_{1,1,j}\Delta S_{1,j}\Delta S_{2,j} + 2\Delta M_{1,2,j}\Delta S_{1,j} + 2\Delta M_{2,1,j}\Delta S_{2,j}). \end{aligned} \tag{24}$$

Applying these realized cumulants allows us to define the third and the fourth comoment swaps described in Table 1. These swaps can be hedged with some securities as Proposition 2 describes.

[Table 1 about here]

Proposition 2. Higher order (co)moment swaps

When we define higher order (co)moment swaps as Table 1 describes, each swap can be replicated with assets S_1, S_2, risk free asset, and the securities that pay $S_{1,T}$, $S_{2,T}$, $S_{1,T}^2$, $S_{1,T}S_{2,T}$, $S_{1,T}^3$, $S_{1,T}^2 S_{2,T}$, or $S_{1,T}S_{2,T}^2$ at time T.

Proof is in the appendix.

[6] Note that the third order joint cumulant of (X_1, X_2, X_3) and the fourth order joint cumulant of (X_1, X_2, X_3, X_4) are $E[X_1X_2X_3]$ and $E[X_1X_2X_3X_4] - E[X_1X_2]E[X_3X_4] - E[X_1X_3]E[X_2X_4] - E[X_1X_4]E[X_2X_3]$ respectively when expectation of each X_i is zero. Therefore, (16) is the third order joint cumulant as well as the third comoment because each S_i is martingale. Similarly, (11) is the third cumulant.

One may wonder about the Aggregation Property for log prices because of log returns' merits like an additivity over time. However when we consider asset allocation among several assets, arithmetic return is more natural than geometric return. So we want to focus on the Aggregation Property with arithmetic returns but a generalized function with the Aggregation Property for log price series is also given as the next propositions. Like in the case of arithmetic return, the Aggregation Property is meaningful when we can find some realized moments. To explore realized moments in a general sense, we define generalized comoments and realized comoments as follows:

Definition 2. A generalized (k,l)-comoment function

We call $f^{k,l}$ a *generalized (k,l)-comoment function* if and only if $f^{k,l}$ is a two dimensional analytic function such that $\frac{f^{k,l}(s_1,s_2)}{s_1^k s_2^l} \to 1$ as $(s_1,s_2) \to (0,0)$. In addition, a *generalized* (k,0)-*comoment function* or a *generalized* (0,k)-*comoment function* is shortly called a *generalized k-moment function* and denoted as one dimensional function f^k.

Definition 3. A realized (k,l)-comoment element

Let $x = (s_1, s_2, m)'$ be a partitioned vector process where m consists of $m_{k,l}$ such that $m_{k,l}(t) = E_t[f^{k,l}(s_1(T) - s_1(t), s_2(T) - s_2(t))]$ for a generalized (k,l)-comoment function $f^{k,l}$. Then, a function g with the Aggregation Property is called a *realized* (k,l)-*comoment element* if and only if it is decomposed as follows

$$g(x_\tau - x_t) = \eta(x_\tau - x_t) + g_r^{k,l}(s_{1,\tau} - s_{1,t}, s_{2,\tau} - s_{2,t}) \tag{25}$$

where η is a function that satisfies $E_t[\eta(x_\tau - x_t)] = 0$ for $t < \tau$ and $g_r^{k,l}$ is a function that satisfies the condition of a generalized (k,l)-comoment function. In addition, a *realized* (*k*,*0*)-*comoment element* or a *realized* (*0*,*k*)-*comoment element* is shortly called a *realized k-moment element*.

As Neuberger shows, there is a realized (3,0)-comoment element with the following decomposition:

$$\begin{aligned} g(\Delta x) &= -12(e^{\Delta s_1}-1)+6\Delta s_1-3\Delta v_1+3e^{\Delta s_1}(\Delta v_1+2\Delta s_1) \\ &= 3\Delta v_1(e^{\Delta s_1}-1)+6(\Delta s_1 e^{\Delta s_1}-2e^{\Delta s_1}+\Delta s_1+2) \end{aligned} \tag{26}$$

However, as shown in the next propositions and corollaries, we cannot get other realized moments under our condition; realized (2,1)-comoment element or realized (4,0)-comoment element does not exist. We finish this section with presenting two propositions and two corollaries. In the propositions, $s_{i,t}$ denotes $\ln(S_{i,t})$ for a martingale process S_i, and $m_{k,l,t}$ denotes $E_t[f_{k,l}(s_{1,T}-s_{1,t}, s_{2,T}-s_{s,t})]$ with a generalized (k,l)-comoment function $f^{k,l}$ or f^k when l=0.

Proposition 3. When $x=(s_1, m_{2,0}, m_{3,0})'$ is a vector valued process on $0 \le t \le T$, an analytic function g has the Aggregation Property on the vector valued process x if and only if, for some constants h_*, g is represented as follows:

$$\begin{aligned} g(s_1, m_{2,0}, m_{3,0}) &= h_1(e^{s_1}-1)+h_2 s_1+h_3 m_{2,0}+h_4 m_{3,0} \\ &\quad + h_5(m_{2,0}+am_{3,0}-2s_1)^2+h_6(m_{2,0}+am_{3,0}+2s_1)e^{s_1} \end{aligned} \tag{27}$$

with one of the following 3 conditions:

i) $h_5=h_6=0$.

ii) $h_6=0$ and $f^2(s)+af^3(s)=2(e^s-s-1)$ for a constant a.

iii) $h_5=0$ and $f^2(s)+af^3(s)=2(se^s-e^s+1)$ for a constant a.

Proof is in the Appendix B.

Corollary 4. When $x=(s_1, m_{2,0}, m_{3,0})'$ is a vector valued process on $0 \le t \le T$, there is no realized 4-moment element.

Proof is in the Appendix B.

Proposition 5. When $x = (s_1, s_2, m_{2,0}, m_{0,2}, m_{1,1})'$ is a vector valued process on $0 \leq t \leq T$, a multidimensional analytic function g has the Aggregation Property on the vector valued process x if and only if, for some constants h_*, g is represented as follows:

$$\begin{aligned} g(s_1, s_2, m_{2,0}, m_{0,2}, m_{1,1}) = h_1(e^{s_1} - 1) + h_2 s_1 + h_3(e^{s_2} - 1) + h_4 s_2 + h_5 m_{2,0} \\ + h_6 m_{0,2} + h_7 m_{1,1} + h_8(m_{2,0} - 2s_1)^2 + h_9(m_{0,2} - 2s_2)^2 \\ + h_{10}(m_{2,0} - 2s_1)(m_{0,2} - 2s_2) + h_{11} e^{s_1}\left(2m_{1,1} - m_{0,2} + 2s_2\right) \\ + h_{12} e^{s_2}\left(2m_{1,1} - m_{2,0} + 2s_1\right) + h_{13} e^{s_1}(m_{2,0} + 2s_1) \\ + h_{14} e^{s_2}(m_{0,2} + 2s_2) \end{aligned} \tag{28}$$

with one of the following 5 conditions:

i) $h_{12} = h_{13} = h_{14} = 0$, $f^{1,1}(s_1, s_2) = s_2(e^{s_1} - 1)$, and $f^2(s) = 2(e^s - s - 1)$.

ii) $h_{11} = h_{13} = h_{14} = 0$, $f^{1,1}(s_1, s_2) = s_1(e^{s_2} - 1)$, and $f^2(s) = 2(e^s - s - 1)$.

iii) $h_{11} = h_{12} = h_{13} = h_{14} = 0$ and $f^2(s) = 2(e^s - s - 1)$.

iv) $h_8 = h_9 = h_{10} = h_{11} = h_{12} = 0$ and $f^2(s) = 2(se^s - e^s + 1)$.

v) $h_8 = h_9 = h_{10} = h_{11} = h_{12} = h_{13} = h_{14} = 0$.

Proof is in the Appendix B.

Corollary 6. When $x = (s_1, s_2, m_{2,0}, m_{0,2}, m_{1,1})'$ is a vector valued process on $0 \leq t \leq T$, there is no realized (2,1)-comoment element.

Proof is similar to the proof of Corollary 4.

III. Practical issues on the estimation

In this section, we investigate estimators of the third and the fourth cumulants for the real data. Since Proposition 1 holds only if each security is martingale, we use forward price for each $S_{i,j}$.[7] Hereafter, each price represents itself normalized by initial price. Accordingly, $S_{i,T}-S_{i,0}$ implies return between time 0 and *T*. By contrast with the price $S_{i,j}$, moments of prices are not directly observed. However, according to Bakshi, Kapadia and Madan (2003), we can get an expectation of a twice-continuously differentiable function of $S_{i,j}$ when there are continuum of European calls and puts. By adopting their method, implied moments of $S_{i,T}-S_{i,j}$ are given as follows:

$$E_j\left[\left(S_{i,T}-S_{i,j}\right)^n\right]=n(n-1)\left(\int_0^{S_{i,j}}(x-S_{i,j})^{n-2}P_j(x)dx+\int_{S_{i,j}}^{\infty}(x-S_{i,j})^{n-2}C_j(x)dx\right),\quad n\geq 2 \quad (29)$$

where $P_j(x)$ is a forward price of a put option at time *j* with an exercise price *x* and a maturity *T*. In addition, $C_j(x)$ is a forward price of a call option defined as $P_j(x)$.

By contrast with the moments of a single security, comoments between securities are not obtainable only with individual European options. Instead, adopting some exotic options makes it possible to get the comoments. For example, when we have continuum of basket options or spread options, Equation (29) with *n*=2 makes us to get $\mathrm{var}_j(S_{1,T}+S_{2,T})$ or $\mathrm{var}_j(S_{1,T}-S_{2,T})$. When it is combined with Equation (30), we can get covariance as follows:

$$\mathrm{cov}_j(S_{1,T},S_{2,T})=\pm\frac{\mathrm{var}_j(S_{1,T}\pm S_{2,T})-\mathrm{var}_j(S_{1,T})-\mathrm{var}_j(S_{2,T})}{2} \quad (30)$$

Similarly, if we have continuum of basket options and spread options in addition to the individual options, we can get third moment of $S_{1,T}\pm S_{2,T}$. Therefore we obtain third comoment due to Equation (31) with $A=S_{1,T}-S_{1,j}$ and $B=S_{2,T}-S_{2,j}$ as follows:

$$E[A^2B]=\frac{E[(A+B)^3]-E[(A-B)^3]-2E[B^3]}{6} \quad (31)$$

Practically, basket options or spread options, composed with individual security and market

[7] Remind that prices are derived through the risk neutral measure while they evolve under the real measure. Therefore both of the implied moment and the realized moment in this paper can be understood as proxies. Bias of the estimate due to the different probability measure is the payoff of hedging strategy of each swap in the proof of Proposition 2. Hereafter, we represent expectations in terms of the risk neutral measure.

index, are not enough to get the implied second and third comoment. However, as Kempf, Korn and Saßning (2015) point, index options are already basket options because indices are portfolio of individual options. Thus following two assumptions make it possible to get covariance between return of individual stock and index. One of them is that asset returns follow an index model with time varying α and β. Then, at each time t_j, conditional distribution between a stock index ($S_{M,T}$) and an each stock price ($S_{i,T,}$) are represented as follows:

$$S_{i,T} = \alpha_{i,j} + \beta_{i,j} S_{M,T} + \varepsilon_{i,j},\ i \in \{1,...,I\},\ 0 \le t_j < t_N = T\ . \tag{32}$$

The second assumption is that ratio of idiosyncratic risk over total risk is same, $1-\rho_j$ with $0 \le \rho_j \le 1$, for all securities at each time t_j. The combination of these two assumptions yields that $\beta_{i,j}^2 V_{i,j} = \rho_j V_{M,j}$ or

$$\beta_{i,j} = \sqrt{\rho_j \frac{V_{i,j}}{V_{M,j}}}\ . \tag{33}$$

where $V_{i,j} = \mathrm{var}_j(S_{i,T})$ and $V_{M,j} = \mathrm{var}_j(S_{M,T})$. Because a beta of the index portfolio is one, $\sum_{i=1}^{I} w_{i,j}\beta_{i,j} = 1$ holds. And this condition yields

$$\rho_j = \frac{V_{M,j}}{\left(\sum_{i=1}^{I} w_{i,j}\sqrt{V_{i,j}}\right)^2}\ . \tag{34}$$

Hence, the covariance between the price and the index are represented as follows:

$$\begin{aligned} C_{i,j} &\equiv \mathrm{cov}_j(S_{i,T}, S_{M,T}) \\ &= \beta_{i,j} V_{M,j} \\ &= \sqrt{\rho_j V_{i,j} V_{i,M}} \\ &= \frac{\sqrt{V_{i,j}}}{\sum_{i=1}^{I} w_{i,j}\sqrt{V_{i,j}}} V_{M,j} \end{aligned} \tag{35}$$

Therefore, a realized third comoment is represented as follows:

$$cTM_{M,M,i}^{real} \equiv \sum_{j=1}^{N} \left\{ \left(\Delta S_{M,j}\right)^2 \Delta S_{i,j} + \Delta S_{i,j}\Delta V_{M,j} + 2\Delta S_{M,j}\Delta C_{i,j} \right\}. \tag{36}$$

Higher order moments are generally provided with standardized form. Among several ways that standardize the third comoment, Kraus and Litzenberger (1976) define coskewness as

$$\gamma_i \equiv \frac{cTM_{M,M,i}}{TM_M} \tag{37}$$

In addition, they show the relation as follows:

$$E[R_i] = \beta_i \lambda_1 + \gamma_i \lambda_2 \tag{38}$$

where R_i is a return of the asset *i*. By adopting the same definition, a realized coskewness is defined as $\gamma_i^{real} \equiv \frac{cTM_{M,M,i}^{real}}{TM_M^{real}}$ with

$$TM_M^{real} \equiv \sum_{j=1}^{N} \left\{ \left(\Delta S_{M,j}\right)^3 + 3\Delta S_{M,j} \Delta V_{M,j} \right\} \tag{39}$$

Additionally one can show that implied gamma at time zero γ_i^{imp} is same with $\beta_{i,0}$ under our assumption, the market model.

IV. Empirical study

The focus of the analysis is two folds. The first part investigates behaviors of the cumulants of S&P 500 returns. And the second is about relations between lagged cumulants of individual stock returns and subsequent returns for the components of Dow Jones Industrial Average (DJIA). For the analysis, implied volatilities, prices of underlying securities, dividends, and risk-free rate from January 1996 to August 2014 are used. We get those of S&P 500, and the components of DJIA from Option Metrics through WRDS. To get continuum of option prices for each strike price, we use the methodology of Carr and Wu (2009) and Neuberger (2012), after options with zero bid price are deleted.

IV.1. Cumulants of the S&P 500 returns

Table 2 shows descriptive statistics of cumulants of the S&P 500 returns. It shows that each realized value is closer to sample values than implied values, in the case of monthly and quarterly returns, although standard deviations of realized values are greater than those of implied values. However, in the case of annual returns, implied values are closer to sample values with small standard deviations. In addition, returns are less negatively skewed and less leptokurtic as time to maturity increases. This pattern could be related to i.i.d. returns, because moment of n-period return is n times moment of 1-period return when return of multi-period is additive and each 1-period return is i.i.d.. However as shown in Table 3, adjusted skewness and adjusted kurtosis are not appears to be constant.[8] It can be from ignored compounding of arithmetic return but it holds even when the compound effect is small; they are different even in the 1-month and 3-month comparison. It implies that sum of nth order returns of sub-periods cannot generate nth order moment of a full period.

[Table 2 about here]

[Table 3 about here]

Since the real probability measure is different from the risk neutral measure, the process of the price is not genuine martingale. Therefore, using the implied second and third moments to get the realized fourth cumulants arises a question of whether the fourth cumulants are reliable. To clarify the validity of lower order implied moments, Table 4 represents time series regression of each realized value on implied and lagged realized values. According to Table 4.A, both implied and lagged realized moments are significant in the univariate regression of the second and the third moments. In addition, implied moments are significant even in the two-variable regression of the second and the third moments while the significances of lagged realized moments vanish. Therefore using the implied second and third moments to yield the realized fourth cumulants is justified in some sense. Now let us deal with standardized moments, which are skewness and kurtosis. Both implied and lagged realized terms are significant in the regression of both skewness and kurtosis. Therefore lagged realized kurtosis provides some information to future realized kurtosis.

The properties of the second and the third moments are also valid in the quarterly and semiannual returns; both the implied and lagged realized moments are significant in the univariate regression, and implied moments are significant in the two variable regression.

[8] The period adjusted skewness is skewness of monthly return which is calculated by sample skewness of return of various period under the i.i.d. returns.

However, they are insignificant in the annual analysis. In addition, implied skewness and kurtosis are significant as like the monthly case. But lagged realized terms are mostly insignificant.

[Table 4 about here]

IV.2. (Joint) cumulants of returns and subsequent returns

This section investigates relations between cumuanlts of returns and subsequent returns on a month-end by month-end basis. Because expirations of the options are not the end of the month, as a proxy, we use interpolated 30-day volatilities of options from volatility surface of Option Metrics for each day. This analysis, based on the end of the month, provides similar results to the analysis based on the expiration of the options. However, this makes it easy to get risk adjusted returns based on Fama and French (1993).

[Table 5 about here]

Table 5 shows average of regression results about comoments. Panel A is about average of time series regression for each security, and Panel B is about average of cross sectional regression for each time. These show that implied comoments have the greatest determinant coefficient among the univariate regressions in the both of time series and cross sectional analysis. In addition, it shows that using the implied covariance to yield the realized third comoment is reasonable.

[Table 6 about here]

Table 6 compares portfolio's return and moments after it is constructed based on implied or realized moments. Panel A represents return, moments, and comoments after it is constructed based on the rank of implied variance. It shows that portfolios keep their order of variance. In other words, a portfolio with the greatest (smallest) implied variance precedes the greatest (smallest) realized variance and the difference between the realized variances is significant. However, the difference of the returns is insignificant. Panel B – Panel G shows the similar results; portfolios keep their order of moments with significant differences but the differences of returns are insignificant. Despite the insignificance of the differences of returns, sizes of the differences are not economically ignorable. For the robustness, Table 7 provides the performances with controlling risk of the other sources while the portfolios in the Table 6 are constructed without controlling risk of the other sources.

[Table 7 about here]

Panel A of Table 7 shows that a portfolio constructed from small implied volatility takes high return. However, it is vague whether the result is based on the idiosyncratic volatility risk solely because implied beta or implied gamma is equivalent to the implied variance under the assumption of market model. To decompose the effects among idiosyncratic volatility, beta, and gamma, we construct the portfolios based on lagged realized moments. Panel D, E, and F show that all of idiosyncratic risk, beta, and gamma are linked to the returns of portfolios; portfolios with low variance, beta, and gamma are along with high return which is consistent with the Panel A. Likewise, other panels show the link between higher order moments and returns of portfolios. Although Panel B and C show insignificant difference of the zero cost portfolios, the size of the return is not economically ignorable. In addition, Panel G and H show that portfolios with low skewness and high kurtosis are along with high return. Hence the results, which use the realized moments, are generally in line with the literature; for example, high beta is linked with low return as Frazzini and Pedersen (2014), and low gamma is linked to high return as Harvey and Siddique (2000), and low skewness or high kurtosis is linked to high return as Amaya et al. (2015) and Conrad, Dittmar and Ghysels (2013).

V. Concluding Remark

In spite of importance of higher-order moments of returns, estimation of realized higher order moments is not as simple as estimation of the second moments. Neuberger (2012) takes advantage of variance process to solve this problem for the third moment. Moreover he shows that there is no higher order realized moments under this information set. As lots of studies emphasize importance of kurtosis or comoments as well as the skewness, we develop the realized fourth cumulant and the realized comoments. Although lack of exotic options limits the accessibility of the comoments, we address a solution about it. In addition, predictability of the implied lower-order comoments about the realized comoments supports the solution of this study.

In addition, we conduct several empirical tests about the realized moments. All the realized moments are explained with implied moments with greater determinant coefficients. It implies both the realized and implied moments are well functioning. Finally, the relations between realized moments of returns and subsequent returns coincide with the literature. Since the realized moments make it possible to understand the distribution of a return of asset for a specific period, we hope this measure to be applied in several studies in the future.

Appendix A: Proofs of Proposition 1 and 2.

Proof of Proposition 1.

Consider processes $S_1(t)$ and $S_2(t)$ for $t \in \{0,1,2\}$ and a moment vector process $\vec{M} = (M_{2,0}, M_{1,1}, M_{0,2}, M_{3,0}, M_{2,1}, M_{1,2}, M_{0,3})$

where

$$M_{k,l}(t) = E_t\left[(S_1(2) - S_1(t))^k (S_2(2) - S_2(t))^l\right]. \tag{A1}$$

In addition, assume

$$(S_1, S_2, \overrightarrow{M}) : (0,0,\vec{m}) \to \begin{cases} (s_{1,1}, s_{2,1}, \vec{\alpha}) & \to & (s_{1,1} + \eta_1, s_{2,1} + \eta_2, \vec{0}) & \Pr = \pi_1 \\ (s_{1,2}, s_{2,2}, \vec{0}) & \to & (s_{1,2}, s_{2,2}, \vec{0}) & \Pr = \pi_2 \\ \vdots & & \vdots & \\ (s_{1,n}, s_{2,n}, \vec{0}) & \to & (s_{1,n}, s_{2,n}, \vec{0}) & \Pr = \pi_n \end{cases} \tag{A2}$$

with $\sum_{j=1}^{n} \pi_j = 1$, $\sum_{j=1}^{n} \pi_j s_{i,j} = 0$, $E[\eta_i] = 0$, $E\left[\eta_1^k \eta_2^l\right] = \alpha_{k,l}$, $\vec{\alpha} = (\alpha_{2,0}, \cdots, \alpha_{0,3})$, $\vec{m} = \vec{M}(0)$, $x_+ = \max(x,0)$ and

$$m_{k,l} = M_{k,l}(0) = \begin{cases} \pi_1 \alpha_{k,l} + \sum_{j=1}^{n} \pi_j s_{1j}^k s_{2,j}^l, & if \;\; k+l=2 \\ \pi_1 (\alpha_{k,l} + l_+ \alpha_{k-1,l} s_{1,1} + k_+ \alpha_{k,l-1} s_{2,1}) + \sum_{j=1}^{n} \pi_j s_{1,j}^k s_{2,j}^l, & if \;\; k+l=3 \end{cases} \tag{A3}$$

The Aggregation Property implies $g(0,...,0) = 0$ and

$$\begin{aligned} & E\left[\pi_1 g\left(s_{1,1} + \eta_1, s_{2,1} + \eta_2, -\vec{m}\right)\right] + \sum_{j=2}^{n} \pi_j g\left(s_{1,j}, s_{2,j}, -\vec{m}\right) \\ & = \pi_1 g\left(s_{1,1}, s_{2,1}, \vec{\alpha} - \vec{m}\right) + E\left[\pi_1 g\left(\eta_1, \eta_2, -\vec{\alpha}\right)\right] + \sum_{j=2}^{n} \pi_j g\left(s_{1,j}, s_{2,j}, -\vec{m}\right) \end{aligned} \tag{A4}$$

or

$$E\left[g\left(s_{1,1} + \eta_1, s_{2,1} + \eta_2, -\vec{m}\right)\right] = g\left(s_{1,1}, s_{2,1}, \vec{\alpha} - \vec{m}\right) + E\left[g\left(\eta_1, \eta_2, -\vec{\alpha}\right)\right]. \tag{A5}$$

Because Equation (A5) holds when $(s_{1,1}, s_{2,1}) = (0,0)$,

$$E\left[g\left(\eta_1, \eta_2, -\vec{m}\right)\right] = g\left(0,0, \vec{\alpha} - \vec{m}\right) + E\left[g\left(\eta_1, \eta_2, -\vec{\alpha}\right)\right]. \tag{A6}$$

Differentiating Equation (A6) with respect to m_{k-2} yields Equation (A7)

$$E\left[g_k\left(\eta_1, \eta_2, -\vec{m}\right)\right] = g_k\left(0,0, \vec{\alpha} - \vec{m}\right) \tag{A7}$$

where g_k is a partial differentiation with respect to the (k-2)th term of the M. i.e. $g_3 = \frac{\partial g}{\partial M_{2,0}}$, $g_4 = \frac{\partial g}{\partial M_{1,1}}$, … , $g_9 = \frac{\partial g}{\partial M_{0,3}}$.

When we substitute $\vec{\alpha} = \vec{m}$ into (A7),

$$E[g_k(\eta_1, \eta_2, -\alpha)] = g_k(0,0,0,0,0) \tag{A8}$$

Therefore, for a constant $a_{k,0}$ and functions $A_{k,*}$, g_k is represented as follows:

$$\begin{aligned} g_k(s_1, s_2, M) &= a_{k,0} + A_{k,1}(M)s_1 + A_{k,2}(M)s_2 + A_{k,3}(M)(s_1^2 + M_{2,0}) \\ &\quad + A_{k,4}(M)(s_1 s_2 + M_{1,1}) + A_{k,5}(M)(s_2^2 + M_{0,2}) + A_{k,6}(M)(s_1^3 + M_{3,0}) \\ &\quad + A_{k,7}(M)(s_1^2 s_2 + M_{2,1}) + A_{k,8}(M)(s_1 s_2^2 + M_{1,2}) + A_{k,9}(M)(s_2^3 + M_{0,3}) \end{aligned} \tag{A9}$$

Substituting $\vec{\alpha}_{-(l-2)} = \vec{m}_{-(l-2)}$ and (A9) into the (A7) yields following:[9]

$$A_{k,l}(-\vec{m})(\alpha_{l-2} - m_{l-2}) = A_{k,l}(0,...0, \alpha_{l-2} - m_{l-2}, 0,..0)(\alpha_{l-2} - m_{l-2}),\ l=3,..,9. \tag{A10}$$

Since π, $s_{i,j}$, α are arbitrary, $A_{k,3}, \cdots, A_{k,9}$ are constants.

$$\begin{aligned} g_k(s_1, s_2, M) &= a_{k,0} + A_{k,1}(M)s_1 + A_{k,2}(M)s_2 + a_{k,3}(s_1^2 + M_{2,0}) \\ &\quad + a_{k,4}(s_1 s_2 + M_{1,1}) + a_{k,5}(s_2^2 + M_{0,2}) + a_{k,6}(s_1^3 + M_{3,0}) \\ &\quad + a_{k,7}(s_1^2 s_2 + M_{2,1}) + a_{k,8}(s_1 s_2^2 + M_{1,2}) + a_{k,9}(s_2^3 + M_{0,3}) \end{aligned} \tag{A11}$$

Now let us simplify $A_{k,1}$ and $A_{k,2}$. Differentiating (A5) with respect to m_{k-2} yields:

$$E\left[g_k\left(s_{1,1} + \eta_1, s_{2,1} + \eta_2, -\vec{m}\right)\right] = g_k\left(s_{1,1}, s_{2,1}, \vec{\alpha} - \vec{m}\right). \tag{A12}$$

Substituting (A11) into (A12) yields

$$\begin{aligned} &-\left(A_{k,1}(\vec{\alpha} - \vec{m}) - A_{k,1}(-\vec{m}) - 3a_{k,6}\alpha_{2,0} - 2a_{k,7}\alpha_{1,1} - a_{k,8}\alpha_{0,2}\right)s_{1,1} \\ &= \left(A_{k,2}(\vec{\alpha} - \vec{m}) - A_{k,2}(-\vec{m}) - a_{k,7}\alpha_{2,0} - 2a_{k,8}\alpha_{1,1} - 3a_{k,9}\alpha_{0,2}\right)s_{2,1} \end{aligned} \tag{A13}$$

Because Equation (A13) is valid for arbitrary α and s,

$$A_{k,1}(M) = 3a_{k,6}M_{2,0} + 2a_{k,7}M_{1,1} + a_{k,8}M_{0,2} + a_{k,1} \tag{A14}$$

$$A_{k,2}(M) = a_{k,7}M_{2,0} + 2a_{k,8}M_{1,1} + 3a_{k,9}M_{0,2} + a_{k,2} \tag{A15}$$

for some $a_{k,1}$ and $a_{k,2}$. Therefore we obtain the following

---

[9] $\vec{\alpha}_{-i}$ represents $\vec{\alpha}$ without the ith element. For example, $\vec{\alpha}_{-1} = (\alpha_{1,1}, \alpha_{0,2}, ..., \alpha_{0,3})$. $\vec{m}_{-i}$ is defined similarly.

$$
\begin{aligned}
g_k(s_1, s_2, M) = a_{k,0} + a_{k,1}s_1 + a_{k,2}s_2 + a_{k,3}(s_1^2 + M_{2,0}) + a_{k,4}(s_1 s_2 + M_{1,1}) \\
+ a_{k,5}(s_2^2 + M_{0,2}) + a_{k,6}(s_1^3 + M_{3,0} + 3M_{2,0}s_1) \\
+ a_{k,7}(s_1^2 s_2 + M_{2,1} + 2M_{1,1}s_1 + M_{2,0}s_2) \\
+ a_{k,8}(s_1 s_2^2 + M_{1,2} + M_{0,2}s_1 + 2M_{1,1}s_2) \\
+ a_{k,9}(s_2^3 + M_{0,3} + 3M_{0,2}s_2)
\end{aligned} \tag{A16}
$$

When $k = 3$, integrating (A16) with respect to M_{k-2} yields Equation (A17).

$$
\begin{aligned}
g(s_1, s_2, M) = a_{3,0}M_{2,0} + a_{3,1}M_{2,0}s_1 + a_{3,2}M_{2,0}s_2 + a_{3,3}(s_1^2 M_{2,0} + \frac{1}{2}M_{2,0}^2) \\
+ a_{3,4}(s_1 s_2 M_{2,0} + M_{1,1}M_{2,0}) + a_{3,5}(s_2^2 M_{2,0} + M_{0,2}M_{2,0}) \\
+ a_{3,6}(s_1^3 M_{2,0} + M_{3,0}M_{2,0} + \frac{3}{2}M_{2,0}^2 s_1) \\
+ a_{3,7}(s_1^2 s_2 M_{2,0} + M_{2,1}M_{2,0} + 2M_{1,1}M_{2,0}s_1 + \frac{1}{2}M_{2,0}^2 s_2) \\
+ a_{3,8}(s_1 s_2^2 M_{2,0} + M_{1,2}M_{2,0} + M_{2,0}M_{0,2}s_1 + 2M_{2,0}M_{1,1}s_2) \\
+ a_{3,9}(M_{2,0}s_2^3 + M_{2,0}M_{0,3} + 3M_{2,0}M_{0,2}s_2) \\
+ A_{3,10}(\vec{M}_{-1})
\end{aligned} \tag{A17}
$$

Similarly, we can get alternative forms of $g(s_1, s_2, M)$ adopting $k = 4...9$. When we combine these forms, $g(s_1, s_2, M)$ can be represented as

$$
g(s_1, s_2, M) = g_M(M; s_1, s_2) + g_s(s_1, s_2) \tag{A18}
$$

where g_M is a multivariate polynomial whose coefficients are (multivariate polynomial) functions of s_1 and s_2 with a condition of $g_M(\vec{0}; s_1, s_2) = 0$ and g_s is a function of s_1 and s_2 with $g_s(0,0) = 0$ because $g(0,0,\vec{0}) = 0$.

Substituting (A18) and $\vec{m} = \vec{\alpha}$ into Equation (A5) and multiplying $2/k^2$ yields:

$$
\begin{aligned}
\frac{2}{k^2}\left(E\left[g_s\left(s_{1,1} + \eta_1, s_{2,1} + \eta_2\right)\right] - g_s(s_{1,1}, s_{2,1}) - E\left[g_s\left(\eta_1, \eta_2\right)\right]\right) \\
= \frac{2}{k^2}\left(E\left[g_M\left(-\vec{\alpha}; s_{1,1} + \eta_1, s_{2,1} + \eta_2\right)\right] - g_M(\vec{0}; s_{1,1}, s_{2,1}) - E\left[g_M\left(-\vec{\alpha}; \eta_1, \eta_2\right)\right]\right)
\end{aligned} \tag{A19}
$$

When we substitute $(\eta_1, \eta_2) = \begin{pmatrix} (k,0) & \Pr = 1/2 \\ (-k,0) & \Pr = 1/2 \end{pmatrix}$ into the (A19), the left hand side of (A19) converges to $\frac{\partial^2 g_s(s_1, s_2)}{\partial s_1^2} - \frac{\partial^2 g_s(0,0)}{\partial s_1^2}$ as $k \to 0$ because $g_s(0,0) = 0$. Therefore

$$
\frac{\partial^2 g_s(s_1, s_2)}{\partial s_1^2} = \frac{\partial^2 g_s(0,0)}{\partial s_1^2} + g_{p1}(s_1, s_2) \tag{A20}
$$

with some polynomial g_{p1} because g_M is a multivariate polynomial. Similarly adopting $(\eta_1,\eta_2)=\begin{pmatrix}(0,k) & \Pr=1/2\\ (0,-k) & \Pr=1/2\end{pmatrix}$ yields

$$\frac{\partial^2 g_s(s_1,s_2)}{\partial s_2^2}=\frac{\partial^2 g_s(0,0)}{\partial s_2^2}+g_{p2}(s_1,s_2) \tag{A21}$$

with some polynomial g_{p2}.

Now consider an alternative form of (A19)

$$\begin{aligned}&\frac{1}{2k_1k_2}\left(E\left[g_s\left(s_{1,1}+\eta_1,s_{2,1}+\eta_2\right)\right]-g_s(s_{1,1},s_{2,1})-E\left[g_s\left(\eta_1,\eta_2\right)\right]\right)\\ &=\frac{1}{2k_1k_2}\left(E\left[g_M\left(-\vec{\alpha};s_{1,1}+\eta_1,s_{2,1}+\eta_2\right)\right]-g_M(\vec{0};s_{1,1},s_{2,1})-E\left[g_M\left(-\vec{\alpha};\eta_1,\eta_2\right)\right]\right)\end{aligned} \tag{A22}$$

Next, when we substitute $(\eta_1,\eta_2)=\begin{pmatrix}(k_1,k_2) & \Pr=1/2\\ (-k_1,-k_2) & \Pr=1/2\end{pmatrix}$ and $(\eta_1,\eta_2)=\begin{pmatrix}(k_1,-k_2) & \Pr=1/2\\ (-k_1,k_2) & \Pr=1/2\end{pmatrix}$ into Equation (A22) and subtract each other we get

$$\frac{\partial^2 g_s(s_1,s_2)}{\partial s_1\partial s_2}=\frac{\partial^2 g_s(0,0)}{\partial s_1\partial s_2}+g_{p3}(s_1,s_2) \tag{A23}$$

as $(k_1,k_2)\to(0,0)$ with some polynomial g_{p3}. (A20), (A21), and (A23) implies that $g(s_1,s_2,M)$ is a polynomial of s_1, s_2, $M_{2,0}$, … and $M_{0,3}$.

Now let us substitute (l_1S_1,l_2S_2) into (S_1,S_2) for the function *g*. Since *g* satisfies the Aggregation Property for any l_1 and l_2, each coefficient of $l_1^{k_1}l_2^{k_2}$ also satisfies the Aggregation Property. Hence, for the coefficients of $l_1^{k_1}l_2^{k_2}$, we can construct a spanning set of functions that has the Aggregation Property and it is represented in Table A1 for $k_1\ge k_2$.

[Table A1 about here]

Note that, in the case of $(k_1,k_2)=(4,0)$, $M_{2,0}^2$ and $M_{2,0}s_1^2$ come together as the form $\left(\frac{1}{2}M_{2,0}^2+M_{2,0}s_1^2\right)$ rather than represented separately. It is due to the form of Equation (A17). Some of the other combined terms are from alternatives of (A17) that are omitted in this paper.

According to Neuberger, s_1, s_1^2, $M_{2,0}$ and $s_1^3+3s_1M_{2,0}$ satisfies the Aggregation Property. In addition, every $M_{i,j}$ also satisfies the Aggregation Property by definition of $M_{i,j}$. Now let us consider a case of $(k_1,k_2)=(2,1)$. Substituting

$$g(s_1,s_2,M)=b_1s_1^2s_2+b_2s_1M_{1,1}+b_3s_2M_{2,0}+b_4M_{2,1} \tag{A24}$$

into Equation (A5) yields

$$b_1(2s_{1,1}\alpha_{1,1} + s_{2,1}\alpha_{2,0}) = b_2 s_{1,1}\alpha_{1,1} + b_3 s_{2,1}\alpha_{2,0} \tag{A25}$$

Therefore $b_2 = 2b_3 = 2b_1$ and $b_3 = b_1$ because α and s are arbitrary numbers. It implies that expression (A26) is a candidate for a function with the Aggregation Property.

$$b_1(s_1^2 s_2 + 2s_1 M_{1,1} + s_2 M_{2,0}) + b_4 M_{2,1} \tag{A26}$$

Similarly we can try for the other pairs of (k_1, k_2) and the result is arranged in the Table A2. Without loss of generality, we can let $(s,t,u,T) = (0,1,2,3)$ in Equation (3) and

$$(S_1(\tau), S_2(\tau)) = \left(\sum_{j=1}^{\tau} R_{1,j}, \sum_{j=1}^{\tau} R_{2,j} \right) \tag{A27}$$

for $\tau = 0,1,2,3$ and $R_{i,j}$ such that $E_{j-1}\left[R_{i,j}\right] = 0$. Then each element of the Table A2 satisfies Equation (4).

■

Proof of Proposition 2.

Consider the securities that pay $S_{1,T}$, $S_{2,T}$, $S_{1,T}^2$, $S_{1,T}S_{2,T}$, $S_{1,T}^3$, $S_{1,T}^2 S_{2,T}$, or $S_{1,T}S_{2,T}^2$ at T. Then the (forward) price of each security at time j is $S_{1,j}$, $S_{2,j}$, $S_{1,j}^2 + M_{2,0,j}$, $S_{1,j}S_{2,j} + M_{1,1,j}$, $S_{1,j}^3 + 3S_{1,j}M_{2,0,j} + M_{3,0,j}$, $S_{1,j}^2 S_{2,j} + 2S_{1,j}M_{1,1,j} + S_{2,j}M_{2,0,j} + M_{2,1,j}$, or $S_{1,j}S_{2,j}^2 + 2S_{2,j}M_{1,1,j} + S_{1,j}M_{0,2,j} + M_{1,2,j}$ respectively.

Equipped with $M_{k,l,N} = 0$ for each k and l, we have the following equality:

$$\begin{aligned}
(S_{1,T} - S_{1,0})^2 (S_{2,T} - S_{2,0}) - \sum_{j=1}^{N} \left((\Delta S_{1,j})^2 \Delta S_{2,j} + \Delta S_{2,j}\Delta M_{2,0,j} + 2\Delta S_{1,j}\Delta M_{1,1,j} \right) \\
= -2\sum_{j=0}^{N-1} (S_{1,j}S_{2,j} - S_{1,0}S_{2,0} - M_{1,1,j})\Delta S_{1,j+1} \\
- \sum_{j=0}^{N-1} (S_{1,j}^2 - S_{1,0}^2 - M_{2,0,j})\Delta S_{2,j+1} \\
+ \sum_{j=0}^{N-1} (S_{2,j} - S_{2,0})\Delta(S_{1,j+1}^2 + M_{2,0,j+1}) \\
+ 2\sum_{j=0}^{N-1} (S_{1,j} - S_{1,0})\Delta(S_{1,j+1}S_{2,j+1} + M_{1,1,j+1})
\end{aligned} \tag{A28}$$

Left hand side of (A28) describes the difference between receiving leg and paying leg of the third comoment swap described in Table 1. In addition, right hand side of Equation (A28) describes strategy of a self-financing portfolio which is managed with securities that pay $S_{1,T}$, $S_{2,T}$, $S_{1,T}^2$, or $S_{1,T}S_{2,T}$ at T. Therefore (A28) shows the fairness and replicability of the third comoment swap.

To deal with the properties about the fourth moment swap and non-zero fourth moment swap, we represent the following equality:

$$
\begin{aligned}
&(S_{1,T}-S_{1,0})^4 - 3a(S_{1,T}-S_{1,0})^2 M_{2,0,0} \\
&\qquad - \sum_{j=1}^{N}\left\{(\Delta S_{1,j})^4 + 6(\Delta S_{1,j})^2 \Delta M_{2,0,j} + 4\Delta S_{1,j}\Delta M_{3,0,j} + 3(\Delta M_{2,0,j})^2\right\} \\
&\qquad = 4\sum_{j=0}^{N-1}(M_{3,0,j} - 3S_{1,j}M_{2,0,j} + \frac{3}{2}aS_{1,0}M_{2,0,0} + S_{1,j}^3 - S_{1,0}^3)\Delta S_{1,j+1} \\
&\qquad + 6\sum_{j=0}^{N-1}(M_{2,0,j} - \frac{1}{2}aM_{2,0,0} - S_{1,j}^2 + S_{1,0}^2)\Delta(S_{1,j+1}^2 + M_{2,0,j+1}) \\
&\qquad + 4\sum_{j=0}^{N-1}(S_{1,j} - S_{1,0})\Delta(S_{1,j+1}^3 + 3S_{1,j+1}M_{2,0,j+1} + M_{3,0,j+1}) \\
&\qquad + 3(1-a)M_{2,0,0}^2
\end{aligned}
\tag{A29}
$$

Left hand side of Equation (A29) with a=0 describes the difference between receiving leg and paying leg of the non-zero fourth moment swap. Since the right hand side of Equation (A29) describes strategy of a self-financing portfolio with initial cost $3M_{2,0,0}^2$ when a=0, we see the fairness and replicability of the non-zero fourth moment swap. Similarly, we can see the properties about the fourth moment swap, non-zero asymmetric fourth comoment swap, asymmetric fourth comoment swap, non-zero symmetric fourth comoment swap, and symmetric fourth comoment swap from Equation (A29) with a=1, Equation (A30) with (a_1,a_2)=(0,0), Equation (A30) with $(a_1,a_2)=(a,1-a)$, Equation (A31) with (a_3,a_4,a_5)=(0,0,0), and Equation (A31) with $(a_3,a_4,a_5)=(a,1-a,1)$ respectively.

$$
\begin{aligned}
&(S_{1,T}-S_{1,0})^3(S_{2,T}-S_{2,0})-3a_1(S_{1,T}-S_{1,0})(S_{2,T}-S_{2,0})M_{2,0,0}-3a_2(S_{1,T}-S_{1,0})^2M_{1,1,0}\\
&\quad-\sum_{j=1}^{N}\left\{(\Delta S_{1,j})^3\Delta S_{2,j}+\Delta S_{2,j}\Delta M_{3,0,j}+3(\Delta S_{1,j})^2\Delta M_{1,1,j}\right\}\\
&\quad-\sum_{j=1}^{N}\left\{3\Delta S_{1,j}\Delta S_{2,j}\Delta M_{2,0,j}+3\Delta M_{2,1,j}\Delta S_{1,j}+3\Delta M_{2,0,j}\Delta M_{1,1,j}\right\}\\
&\quad=3\sum_{j=0}^{N-1}\begin{pmatrix}M_{2,1,j}-2S_{1,j}M_{1,1,j}-S_{2,j}M_{2,0,j}+2a_2S_{1,0}M_{1,1,0}\\+a_1S_{2,0}M_{2,0,0}+S_{1,j}^2S_{2,j}-S_{1,0}^2S_{2,0}\end{pmatrix}\Delta S_{1,j+1}\\
&\quad+\sum_{j=0}^{N-1}(M_{3,0,j}-3S_{1,j}M_{2,0,j}+3a_1S_{1,0}M_{2,0,0}+S_{1,j}^3-S_{1,0}^3)\Delta S_{2,j+1}\\
&\quad+3\sum_{j=0}^{N-1}(M_{1,1,j}-a_2M_{1,1,0}-S_{1,j}S_{2,j}+S_{1,0}S_{2,0})\Delta(S_{1,j+1}^2+M_{2,0,j+1})\\
&\quad+3\sum_{j=0}^{N-1}(M_{2,0,j}-a_1M_{2,0,0}-S_{1,j}^2+S_{1,0}^2)\Delta(S_{1,j+1}S_{2,j+1}+M_{1,1,j+1})\\
&\quad+\sum_{j=0}^{N-1}(S_{2,j}-S_{2,0})\Delta(S_{1,j+1}^3+3S_{1,j+1}M_{2,0,j+1}+M_{3,0,j+1})\\
&\quad+3\sum_{j=0}^{N-1}(S_{1,j}-S_{1,0})\Delta\begin{pmatrix}S_{1,j+1}^2S_{2,j+1}+2S_{1,j+1}M_{1,1,j+1}\\+S_{2,j+1}M_{2,0,j+1}+M_{2,1,j+1}\end{pmatrix}\\
&\quad+3(1-a_1-a_2)M_{2,0,0}M_{1,1,0}
\end{aligned}
\qquad \text{(A30)}
$$

$$
\begin{aligned}
&(S_{1,T}-S_{1,0})^2(S_{2,T}-S_{2,0})^2-a_3(S_{1,T}-S_{1,0})^2 M_{0,2,0} \\
&\qquad -a_4(S_{2,T}-S_{2,0})^2 M_{2,0,0}-2a_5(S_{1,T}-S_{1,0})(S_{2,T}-S_{2,0})M_{1,1,0} \\
&\qquad -\sum_{j=1}^{N}\left\{((\Delta S_{1,j})^2+\Delta M_{2,0,j})((\Delta S_{2,j})^2+\Delta M_{0,2,j})+2(\Delta M_{1,1,j})^2\right\} \\
&\qquad -\sum_{j=1}^{N}\left\{4\Delta M_{1,1,j}\Delta S_{1,j}\Delta S_{2,j}+2\Delta M_{1,2,j}\Delta S_{1,j}+2\Delta M_{2,1,j}\Delta S_{2,j}\right\} \\
&\qquad\quad =2\sum_{j=0}^{N-1}\begin{pmatrix} M_{1,2,j}-2S_{2,j}M_{1,1,j}-S_{1,j}M_{0,2,j}+a_5S_{2,0}M_{1,1,0} \\ +a_3S_{1,0}M_{0,2,0}+S_{1,j}S_{2,j}^2-S_{1,0}S_{2,0}^2 \end{pmatrix}\Delta S_{1,j+1} \\
&\qquad\quad +2\sum_{j=0}^{N-1}\begin{pmatrix} M_{2,1,j}-2S_{1,j}M_{1,1,j}-S_{2,j}M_{2,0,j}+a_5S_{1,0}M_{1,1,0} \\ +a_4S_{2,0}M_{2,0,0}+S_{1,j}^2S_{2,j}-S_{1,0}^2S_{2,0} \end{pmatrix}\Delta S_{2,j+1} \\
&\qquad\quad +\sum_{j=0}^{N-1}(M_{0,2,j}-a_3M_{0,2,0}-S_{2,j}^2+S_{2,0}^2)\Delta(S_{1,j+1}^2+M_{2,0,j+1}) \\
&\qquad\quad +\sum_{j=0}^{N-1}(M_{2,0,j}-a_4M_{2,0,0}-S_{1,j}^2+S_{1,0}^2)\Delta(S_{2,j+1}^2+M_{0,2,j+1}) \\
&\qquad\quad +4\sum_{j=0}^{N-1}\begin{pmatrix} M_{1,1,j}-\dfrac{1}{2}a_5M_{1,1,0} \\ -S_{1,j}S_{2,j}+S_{1,0}S_{2,0} \end{pmatrix}\Delta(S_{1,j+1}S_{2,j+1}+M_{1,1,j+1}) \\
&\qquad\quad +2\sum_{j=0}^{N-1}(S_{2,j}-S_{2,0})\Delta\begin{pmatrix} S_{1,j+1}^2S_{2,j+1}+2S_{1,j+1}M_{1,1,j+1} \\ +S_{2,j+1}M_{2,0,j+1}+M_{2,1,j+1} \end{pmatrix} \\
&\qquad\quad +2\sum_{j=0}^{N-1}(S_{1,j}-S_{1,0})\Delta\begin{pmatrix} S_{2,j+1}^2S_{1,j+1}+2S_{2,j+1}M_{1,1,j+1} \\ +S_{1,j+1}M_{0,2,j+1}+M_{1,2,j+1} \end{pmatrix} \\
&\qquad\quad +(1-a_3-a_4)M_{2,0,0}M_{0,2,0}+2(1-a_5)M_{1,1,0}^2
\end{aligned}
\tag{A31}
$$

■

Appendix B: Proofs of Proposition 3 and 5 and Corollary 4.

Common property B.

Consider processes $S_1(t)$ and $S_2(t)$ for $t \in \{0,1,2\}$ and moment vector processes $\vec{M} = (M_{2,0}, M_{1,1}, M_{0,2}, M_{3,0}, M_{2,1}, M_{1,2}, M_{0,3})$

In addition assume

$$(S_1, S_2, \overrightarrow{M}) : (0,0,\vec{m}) \rightarrow \begin{cases} (s_{1,1}, s_{2,1}, \vec{\alpha}) & \rightarrow & (s_{1,1}+\eta_1, s_{2,1}+\eta_2, \vec{0}) & \Pr = \pi_1 \\ (s_{1,2}, s_{2,2}, \vec{0}) & \rightarrow & (s_{1,2}, s_{2,2}, \vec{0}) & \Pr = \pi_2 \\ \vdots & & \vdots & \\ (s_{1,n}, s_{2,n}, \vec{0}) & \rightarrow & (s_{1,n}, s_{2,n}, \vec{0}) & \Pr = \pi_n \end{cases} \tag{B1}$$

with $\sum_{j=1}^{n} \pi_j = 1$, $\sum_{j=1}^{n} \pi_j \exp(s_{i,j}) = 0$, $E[\exp(\eta_i)] = 1$, $E\left[f^{k,l}(\eta_1, \eta_2)\right] = \alpha_{k,l}$, $\vec{\alpha} = (\alpha_{2,0}, \cdots, \alpha_{0,3})$, and $\vec{m} = \vec{M}(0)$

where

$$m_{k,l} = \pi_1 E[f^{k,l}(s_{1,1}+\eta_1, s_{2,1}+\eta_2)] + \sum_{j=2}^{n} \pi_j f^{k,l}(s_{1j}, s_{2j}) \tag{B2}$$

and $f^{k,l}$ is a generalized moment function such that $f^{k,l}(0,0) = 0$ and $\lim_{(a,b)\to(0,0)} \frac{f^{k,l}(a,b)}{a^k b^l} = 1$, $f^{l,k}(a,b) = f^{k,l}(b,a)$, and $f^k(a) = f^{k,0}(a,b)$.

Again, $g(0,\ldots,0)=0$ and Equations (A5) - (A12) hold here. Let us represent some of them:

$$E\left[g\left(s_{1,1}+\eta_1, s_{2,1}+\eta_2, -\vec{m}\right)\right] = g\left(s_{1,1}, s_{2,1}, \vec{\alpha}-\vec{m}\right) + E\left[g\left(\eta_1, \eta_2, -\vec{\alpha}\right)\right] \tag{B3}$$

$$E\left[g_k\left(\eta_1, \eta_2, -\vec{m}\right)\right] = g_k\left(0,0,\vec{\alpha}-\vec{m}\right) \tag{B4}$$

$$\begin{aligned} g_k(s_1, s_2, M) = {} & a_{k,0} + A_{k,1}(M)(e^{s_1}-1) + A_{k,2}(M)(e^{s_2}-1) + a_{k,3}(f^2(s_1) + M_{2,0}) \\ & + a_{k,4}(f^{1,1}(s_1,s_2) + M_{1,1}) + a_{k,5}(f^2(s_2) + M_{0,2}) + a_{k,6}(f^3(s_1) + M_{3,0}) \\ & + a_{k,7}(f^{2,1}(s_1,s_2) + M_{2,1}) + a_{k,8}(f^{2,1}(s_2,s_1) + M_{1,2}) + a_{k,9}(f^3(s_2) + M_{0,3}) \end{aligned} \tag{B5}$$

$$E\left[g_k\left(s_{1,1}+\eta_1, s_{2,1}+\eta_2, -\vec{m}\right)\right] = g_k\left(s_{1,1}, s_{2,1}, \vec{\alpha}-\vec{m}\right) \tag{B6}$$

Substituting (B5) into (B6) and differentiating with respect to m_l yields

$$\frac{\partial A_{k,1}(-\vec{m})}{\partial m_l} = \frac{\partial A_{k,1}(\vec{a}-\vec{m})}{\partial m_l}, \quad \frac{\partial A_{k,2}(-\vec{m})}{\partial m_l} = \frac{\partial A_{k,2}(\vec{a}-\vec{m})}{\partial m_l}, \quad l=3,..9 \tag{B7}$$

Therefore each $A_{k,*}(M)$ is an affine function. Accordingly (B5) is represented as follows:

$$\begin{aligned}
g_k(s_1,s_2,M) = a_{k,0} &+ (b_{k,0} + b_{k,1}M_{2,0} + b_{k,2}M_{1,1} + \ldots + b_{k,7}M_{0,3})(e^{s_1}-1) \\
&+ (c_{k,0} + c_{k,1}M_{2,0} + c_{k,2}M_{1,1} + \ldots + c_{k,7}M_{0,3})(e^{s_2}-1) \\
&+ a_{k,3}(f^2(s_1) + M_{2,0}) + a_{k,4}(f^{1,1}(s_1,s_2) + M_{1,1}) + a_{k,5}(f^2(s_2) + M_{0,2}) \\
&+ a_{k,6}(f^3(s_1) + M_{3,0}) + a_{k,7}(f^{2,1}(s_1,s_2) + M_{2,1}) \\
&+ a_{k,8}(f^{2,1}(s_2,s_1) + M_{1,2}) + a_{k,9}(f^3(s_2) + M_{0,3})
\end{aligned} \tag{B8}$$

Proof of Proposition 3.

We use the whole Common property B above. Instead, let us omit all terms related the second security. Accordingly, ignore the $S_2(t)$ and restrict the M as $M=(M_2,M_3)$ with $M_2 = M_{2,0}$ and $M_3 = M_{3,0}$. Then integrating (B8) with respect to M_2 yields

$$\begin{aligned}
g(s_1,M_2,M_3) = a_{1,0}M_2 &+ (b_{1,0}M_2 + b_{1,1}M_2^2/2 + b_{1,3}M_3M_2)(e^{s_1}-1) \\
&+ a_{1,3}(M_2 f^2(s_1) + M_2^2/2) + a_{1,6}M_2(f^3(s_1) + M_3) \\
&+ g^1(s_1,M_3)
\end{aligned} \tag{B9}$$

Similarly, we can get alternative form of (B9) by integrating (B8) with respect to M_3. By combining (B9) and the alternative form, we obtain the following form

$$\begin{aligned}
g(s,M_2,M_3) = a_1M_2 + a_2M_3 &+ (a_3M_2 + a_4M_3 + a_5M_2^2 + a_6M_2M_3 + a_7M_3^2)(e^s-1) \\
&+ a_8(M_2^2 + 2M_2 f^2(s)) + a_9(M_3^2 + 2M_3 f^3(s)) \\
&+ 2a_{10}(M_2M_3 + f^2(s)M_3 + f^3(s)M_2) + g^s(s)
\end{aligned} \tag{B10}$$

for some constants a_* and a function $g^s(*)$ such that $g^s(0)=0$. Substituting (B10) and $\eta_p = \begin{cases} \eta, & \Pr = p \\ 0, & \Pr = 1-p \end{cases}$ into (B3) yields

$$\begin{aligned}
0 = p &\left(\begin{aligned}
&(e^{s_1}-1)\big(a_3\alpha_2 + a_4\alpha_3 - 2a_5m_2\alpha_2 + a_6(-m_2\alpha_3 - m_3\alpha_2) - 2a_7m_3\alpha_3\big) \\
&+ 2a_8\big((f^2(s_1+\eta) - f^2(s_1) - \alpha_2)m_2 + \alpha_2 f^2(s_1)\big) \\
&+ 2a_9\big((f^3(s_1+\eta) - f^3(s_1) - \alpha_3)m_3 + \alpha_3 f^3(s_1)\big) \\
&+ 2a_{10}\left(\begin{aligned} &(f^2(s_1+\eta) - f^2(s_1) - \alpha_2)m_3 + \alpha_3 f^2(s_1) \\ &+ (f^3(s_1+\eta) - f^3(s_1) - \alpha_3)m_2 + \alpha_2 f^3(s_1) \end{aligned} \right) \\
&- E[g^s(s_1+\eta)] + g^s(s_1) + E[g^s(\eta)]
\end{aligned} \right) \\
&+ p^2(e^{s_1}-1)\big(a_5\alpha_2^2 + a_6\alpha_2\alpha_3 + a_7\alpha_3^2\big)
\end{aligned} \tag{B11}$$

Because we can set p arbitrary, coefficients of p and p^2 are zero. Therefore,

$$a_5\alpha_2^2 + a_6\alpha_2\alpha_3 + a_7\alpha_3^2 = 0\,. \tag{B12}$$

It implies that $a_5 = a_6 = a_7 = 0$ because η is arbitrary function with $E\left[e^{\eta}\right]=1$. Accordingly we have the following relation:

$$\begin{aligned} 0 = (e^{s_1} - 1)&\left(a_3\alpha_2 + a_4\alpha_3\right) \\ &+ 2a_8\Big((f^2(s_1+\eta) - f^2(s_1) - \alpha_2)m_2 + \alpha_2 f^2(s_1)\Big) \\ &+ 2a_9\Big((f^3(s_1+\eta) - f^3(s_1) - \alpha_3)m_3 + \alpha_3 f^3(s_1)\Big) \\ &+ 2a_{10}\begin{pmatrix} (f^2(s_1+\eta) - f^2(s_1) - \alpha_2)m_3 + \alpha_3 f^2(s_1) \\ +(f^3(s_1+\eta) - f^3(s_1) - \alpha_3)m_2 + \alpha_2 f^3(s_1) \end{pmatrix} \\ &- E[g^s(s_1+\eta)] + g^s(s_1) + E[g^s(\eta)] \end{aligned} \tag{B13}$$

In addition, coefficients of m_2 and m_3 are zero because we can set them arbitrary.

$$0 = a_8(f^2(s_1+\eta) - f^2(s_1) - \alpha_2) + a_{10}(f^3(s_1+\eta) - f^3(s_1) - \alpha_3) \tag{B14}$$

$$0 = a_{10}(f^2(s_1+\eta) - f^2(s_1) - \alpha_2) + a_9(f^3(s_1+\eta) - f^3(s_1) - \alpha_3) \tag{B15}$$

We have three cases that satisfy both (B14) and (B15).

Condition B.1

i) $a_8 = a_9 = a_{10} = 0$,

ii) $\exists f^3$ such that $\forall(s_1,\eta),\ f^3(s_1+\eta) - f^3(s_1) - \alpha_3 = 0$ and $a_8 = a_{10} = 0$

iii) $\exists(a, f^2, f^3)$ s.t. $\forall(s_1,\eta),\ (f^2(s_1+\eta) - f^2(s_1) - \alpha_2) + a(f^3(s_1+\eta) - f^3(s_1) - \alpha_3) = 0$ with $a_{10} = a_8 a$ and $a_9 = a^2 a_8$.

First, to check about the condition B1.ii), substituting $\eta = \begin{cases} \log(1+k), & \Pr = 0.5 \\ \log(1-k), & \Pr = 0.5 \end{cases}$ into $\frac{2}{k^2}\left(f^3(s_1+\eta) - f^3(s_1) - \alpha_3\right) = 0$ and taking the limit for $k \to 0$ yields:

$$f^{3}{}''(s_1) - f^{3}{}'(s_1) = 0 \tag{B16}$$

Then $f^3(s_1) = b_1 s_1 + b_2 e^{s_1}$ is a solution of (B16) for some b_1 and b_2. However, then f^3 cannot be a generalized (3,0)-comoment function. It implies that the condition B1.ii) is impossible. Now let us check the condition B1.iii).

Let $f^a(x) = f^2(x) + af^3(x)$ and $\eta = \begin{cases} \log(1+k), & \Pr = 0.5 \\ \log(1-k), & \Pr = 0.5 \end{cases}$. Then like the method above, we can show that

$$f^{a}{}''(s_1) - f^{a}{}'(s_1) - 2 = 0 \tag{B17}$$

Since f^2 and f^3 are a generalized (2,0)-comoment function and a generalized (3,0)-comoment function respectively, we have the following solution

$$f^2(s) = 2(e^s - s - 1) - af^3(s) \tag{B18}$$

for some generalized (3,0)-comoment function f^3. Therefore (B13) is arranged as follows:

$$\begin{aligned} E[g^s(s_1+\eta)] - g^s(s_1) - E[g^s(\eta)] = {} & (e^{s_1} - 1)(a_3\alpha_2 + a_4\alpha_3) \\ & + 2a_8(\alpha_2 + a\alpha_3)(f^2(s_1) + af^3(s_1)) \end{aligned} \tag{B19}$$

Again, by letting $\eta = \begin{cases} \log(1+k), & \Pr = 0.5 \\ \log(1-k), & \Pr = 0.5 \end{cases}$ and taking limit, g^s is represented as follows:

$$g^s(s) = a_9 s + a_{10}(e^s - 1) + 4a_8 s^2 + (8a_8 + 2a_3)se^2 \tag{B20}$$

$$\begin{aligned} g(s, M_2, M_3) = {} & a_1M_2 + a_2M_3 + (a_3M_2 + a_4M_3)(e^s - 1) \\ & + a_8((M_2 + aM_3)^2 + 2(M_2 + aM_3)(f^2(s) + af^3(s))) \\ & + a_9 s + a_{10}(e^s - 1) + 4a_8 s^2 + (8a_8 + 2a_3)se^2 \end{aligned} \tag{B21}$$

Then, (B18) implies that

$$\begin{aligned} g(s, M_2, M_3) = {} & a_1M_2 + a_2M_3 + (a_3M_2 + a_4M_3)(e^s - 1) \\ & + a_8(M_2 + aM_3 - 2s)^2 + 4a_8(M_2 + aM_3)(e^s - 1) \\ & + a_9 s + a_{10}(e^s - 1) + (8a_8 + 2a_3)se^s \end{aligned} \tag{B22}$$

or

$$\begin{aligned} g(s, M_2, M_3) = {} & d_1M_2 + d_2M_3 + d_3M_3e^s + d_4(M_2 + aM_3 - 2s)^2 \\ & + d_5(M_2 + aM_3 + 2s)e^s + d_6 s + d_7(e^s - 1) \end{aligned} \tag{B23}$$

where $d_1 = a_1 - a_3$, $d_2 = a_2 - a_4$, $d_3 = a_4 - aa_3$, $d_4 = a_8$, $d_5 = a_3 + 4a_8$, $d_6 = a_9$, and $d_7 = a_{10}$.[10] Then substituting it into (B3) yields

[10] Note that form (B23) include the condition B.1.i).

$$
\begin{aligned}
& d_4(4s_1 + 2(m_2 + am_3 - \alpha_2 - a\alpha_3))(E[2\eta] + \alpha_2 + a\alpha_3) \\
& + (e^{s_1} - 1)\big(d_5(E[2\eta e^{\eta}] - \alpha_2 - a\alpha_3) - d_3\alpha_3\big) = 0
\end{aligned} \tag{B24}
$$

Since s_1 is arbitrary, we have the following cases

Condition B.2

i) $d_3 = d_4 = d_5 = 0$

ii) $d_3 = d_5 = 0$ & $E[2\eta] + \alpha_2 + a\alpha_3 = 0$

iii) $d_4 = 0$ & $E[2\eta e^{\eta}] - \alpha_2 - h\alpha_3 = 0$ with $h = a + d_3 / d_5$

When the Condition B.2.ii) holds, $f^2(s) + af^3(s) = 2(e^s - s - 1)$. And because iii) and ii) are exclusive, Condition B.2.iii) is equivalent to $d_3 = d_4 = 0$ with

$$
E[2\eta e^{\eta}] - \alpha_2 - a\alpha_3 = 0. \tag{B25}
$$

(B25) is equivalent to

$$
f^2(s) + af^3(s) = 2(se^s - e^s + 1) \tag{B26}
$$

Arranging all above yields the equation and the condition of Proposition 3. Without loss of generality, we can let $(s, t, u, T) = (0,1,2,3)$ in Equation (3) and

$$
S(\tau) = \exp\left(\sum_{j=1}^{\tau} r_j\right) \tag{B27}
$$

for $\tau = 0,1,2,3$ and r_j such that $E_{j-1}\left[\exp(r_j)\right] = 1$. Then g above satisfies Equation (4).

■

Proof of Corollary 4.

If a function is a realized (4,0)-comoment element, it should be decomposed as

$$
g(s_1, m_{2,0}, m_{3,0}) = (e^{s_1} - 1)\eta(s_1, m_{2,0}, m_{3,0}) + g^r(s_1)
$$

such that $g^r(s_1) = O(s_1^4)$ because of the restriction, $E\left[e^{\Delta s_1}\right] = 1$. Now let us investigate the each condition. At the first condition, if h_3 or h_4 are not zero, they cannot be eliminated. Therefore, $h_3 = h_4 = 0.$ However, $h_1(e^{s_1} - 1) + h_2 s_1$ is at most $O(s_1^2)$ as $s_1 \to 0$.

At the second condition, if h_5 is zero, it is a case of the first condition. Therefore, it suffices to show the case of nonzero h_5. However, if h_5 is not zero, $m_{2,0}^2$ is not eliminated with zero expectation.

Similarly, at the third condition, investigating nonzero h_6 is enough. However, if h_6 is not zero, To eliminate $m_{2,0}$ and $m_{3,0}$, $h_3 = -h_6$ and $h_4 = -ah_6$. And then, the remaining term is at most $O(s_1^3)$ as $s_1 \to 0$.

■

Proof of Proposition 5.

Proof of Proposition 5 is similar to Proof of Proposition 3. For a convenience, let us replace some notations. At the Common property B, let us restrict the $M = (V_1, V_2, V_c)$ with $V_1 = M_{2,0}$, $V_2 = M_{0,2}$, and $V_c = M_{1,1}$. In addition, f and f_c replace f^2 and $f^{1,1}$ respectively. Then integrating (B8) with respect to V_1 yields

$$\begin{aligned} g(s_1, s_2, V_1, V_2, V_c) = a_{1,0}V_1 &+ (b_{1,0}V_1 + b_{1,1}V_1^2/2 + b_{1,2}V_1V_c + b_{1,3}V_1V_2)(e^{s_1} - 1) \\ &+ (c_{1,0}V_1 + c_{1,1}V_1^2/2 + c_{1,2}V_1V_c + c_{1,3}V_1V_2)(e^{s_2} - 1) \\ &+ a_{1,3}(V_1 f^2(s_1) + V_1^2/2) + a_{1,4}V_1(f^{1,1}(s_1, s_2) + V_c) \\ &+ a_{1,5}V_1(f^2(s_2) + V_2) + g^1(s_1, s_2, V_2, V_c) \end{aligned} \tag{B28}$$

Similarly, we can get alternative form of (B28) by integrating (B8) with respect to V_2 or V_c. By combining (B28) and the alternatives, we obtain the following form

$$\begin{aligned} g(s_1, s_2, V_1, V_2, V_c) = b_0V_c + b_1V_1 + b_2V_2 &+ (e^{s_1} - 1)(b_3V_c + b_4V_1 + b_5V_2 + b_6V_cV_1 \\ &+ b_7V_cV_2 + b_8V_1V_2 + b_9V_c^2 + b_{10}V_1^2 + b_{11}V_2^2) \\ &+ (e^{s_2} - 1)(b_{12}V_c + b_{13}V_1 + b_{14}V_2 + b_{15}V_cV_1 \\ &+ b_{16}V_cV_2 + b_{17}V_1V_2 + b_{18}V_c^2 + b_{19}V_1^2 + b_{20}V_2^2) \\ &+ b_{21}(f(s_1)V_c + V_1V_c + V_1 f_c(s_1, s_2)) \\ &+ b_{22}(f(s_2)V_c + V_2V_c + f_c(s_1, s_2)V_2) \\ &+ b_{23}(f(s_2)V_1 + V_1V_2 + f(s_1)V_2) \\ &+ b_{24}(2f(s_1, s_2) + V_c)V_c + b_{25}(2f(s_1) + V_1)V_1 \\ &+ b_{26}(2f(s_2) + V_2)V_2 + g^s(s_1, s_2) \end{aligned} \tag{B29}$$

with $g^s(0,0) = 0$. Let us substitute (B29) into equation (B3). Then

It is simplified as:

$$
\begin{aligned}
0 = &\left(e^{s_{11}}-1\right)\begin{pmatrix} b_3\alpha_c + b_4\alpha_1 + b_5\alpha_2 + b_6(\alpha_c\alpha_1 - \alpha_1 v_c - \alpha_c v_1) \\ + b_7(\alpha_2\alpha_c - \alpha_2 v_c - \alpha_c v_2) + b_8(\alpha_1\alpha_2 - \alpha_2 v_1 - \alpha_1 v_2) \\ + b_9(\alpha_c^2 - 2\alpha_c v_c) + b_{10}(\alpha_1^2 - 2\alpha_1 v_1) + b_{11}(\alpha_2^2 - 2\alpha_2 v_2) \end{pmatrix} \\
&+ \left(e^{s_2}-1\right)\begin{pmatrix} b_{12}\alpha_c + b_{13}\alpha_1 + b_{14}\alpha_2 + b_{15}(\alpha_c\alpha_1 - \alpha_1 v_c - \alpha_c v_1) \\ + b_{16}(\alpha_2\alpha_c - \alpha_2 v_c - \alpha_c v_2) + b_{17}(\alpha_1\alpha_2 - \alpha_2 v_1 - \alpha_1 v_2) \\ + b_{18}(\alpha_c^2 - 2\alpha_c v_c) + b_{19}(\alpha_1^2 - 2\alpha_1 v_1) + b_{20}(\alpha_2^2 - 2\alpha_2 v_2) \end{pmatrix} \\
&+ b_{21}\begin{pmatrix} (E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_c + f(s_{11})\alpha_c + \alpha_1 f_c(s_{11}, s_{21}) \\ + (E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c)v_1 \end{pmatrix} \\
&+ b_{22}\begin{pmatrix} (E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_c + f(s_{21})\alpha_c + \alpha_2 f_c(s_{11}, s_{21}) \\ + (E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c)v_2 \end{pmatrix} \\
&+ b_{23}\begin{pmatrix} (E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_2 + f(s_{21})\alpha_1 + f(s_{11})\alpha_2 \\ + (E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_1 \end{pmatrix} \\
&+ 2b_{24}((E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c)v_c + f_c(s_{11}, s_{21})\alpha_c) \\
&+ 2b_{25}((E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_1 + f(s_{11})\alpha_1) \\
&+ 2b_{26}((E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_2 + f(s_{21})\alpha_2) \\
&- E[g^s(s_{11}+\eta_1, s_{21}+\eta_2)] + g^s(s_1, s_2) + E[g^s(\eta_1, \eta_2)]
\end{aligned}
\tag{B30}
$$

Let

$$
(\eta_1^p, \eta_2^p) = \begin{cases} (\eta_1, \eta_2) & \Pr = p \\ (0,0) & \Pr = 1-p \end{cases}
\tag{B31}
$$

Then, for $i \in \{1,2\}$, $E\left[e^{\eta_i^p}\right] = 1$, $E[f(\eta_i^p)] = \alpha_i p$ and $E[f_c(\eta_1^p, \eta_2^p)] = \alpha_c p$. Therefore, substituting (η_1^p, η_2^p) into (η_1, η_2) of the previous equation yields

$$
\begin{aligned}
0 = &\, p\left(e^{s_{11}}-1\right)\begin{pmatrix} b_3\alpha_c + b_4\alpha_1 + b_5\alpha_2 + b_6(\alpha_c\alpha_1 p - \alpha_1 v_c - \alpha_c v_1) \\ + b_7(\alpha_2\alpha_c p - \alpha_2 v_c - \alpha_c v_2) + b_8(\alpha_1\alpha_2 p - \alpha_2 v_1 - \alpha_1 v_2) \\ + b_9(\alpha_c^2 p - 2\alpha_c v_c) + b_{10}(\alpha_1^2 p - 2\alpha_1 v_1) + b_{11}(\alpha_2^2 p - 2\alpha_2 v_2) \end{pmatrix} \\
&+ p\left(e^{s_2}-1\right)\begin{pmatrix} b_{12}\alpha_c + b_{13}\alpha_1 + b_{14}\alpha_2 + b_{15}(\alpha_c\alpha_1 p - \alpha_1 v_c - \alpha_c v_1) \\ + b_{16}(\alpha_2\alpha_c p - \alpha_2 v_c - \alpha_c v_2) + b_{17}(\alpha_1\alpha_2 p - \alpha_2 v_1 - \alpha_1 v_2) \\ + b_{18}(\alpha_c^2 p - 2\alpha_c v_c) + b_{19}(\alpha_1^2 p - 2\alpha_1 v_1) + b_{20}(\alpha_2^2 p - 2\alpha_2 v_2) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+pb_{21}\begin{pmatrix}(E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)v_c+f(s_{11})\alpha_c+\alpha_1 f_c(s_{11},s_{21})\\ +(E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)]-f_c(s_{11},s_{21})-\alpha_c)v_1\end{pmatrix}\\
&+pb_{22}\begin{pmatrix}(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)v_c+f(s_{21})\alpha_c+\alpha_2 f_c(s_{11},s_{21})\\ +(E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)]-f_c(s_{11},s_{21})-\alpha_c)v_2\end{pmatrix}\\
&+pb_{23}\begin{pmatrix}(E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)v_2+f(s_{21})\alpha_1+f(s_{11})\alpha_2\\ +(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)v_1\end{pmatrix}\\
&+2pb_{24}((E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)]-f_c(s_{11},s_{21})-\alpha_c)v_c+f_c(s_{11},s_{21})\alpha_c)\\
&+2pb_{25}((E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)v_1+f(s_{11})\alpha_1)\\
&+2pb_{26}((E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)v_2+f(s_{21})\alpha_2)\\
&\qquad -pE[g^s(s_{11}+\eta_1,s_{21}+\eta_2)]+pg^s(s_1,s_2)+pE[g^s(\eta_1,\eta_2)]
\end{aligned}
\tag{B32}
$$

Because (B32) holds for arbitrary p, the coefficient of p^2 should be zero.

$$
\begin{aligned}
0&=\left(e^{s_{11}}-1\right)\left(b_6\alpha_c\alpha_1+b_7\alpha_2\alpha_c+b_8\alpha_1\alpha_2+b_9\alpha_c^2+b_{10}\alpha_1^2+b_{11}\alpha_2^2\right)\\
&+\left(e^{s_2}-1\right)\left(b_{15}\alpha_c\alpha_1+b_{16}\alpha_2\alpha_c+b_{17}\alpha_1\alpha_2+b_{18}\alpha_c^2+b_{19}\alpha_1^2+b_{20}\alpha_2^2\right)
\end{aligned}
\tag{B33}
$$

Since s_{11} and s_{21} are arbitrary, the following holds:

$$
\begin{aligned}
0&=b_6\alpha_c\alpha_1+b_7\alpha_2\alpha_c+b_8\alpha_1\alpha_2+b_9\alpha_c^2+b_{10}\alpha_1^2+b_{11}\alpha_2^2\\
0&=b_{15}\alpha_c\alpha_1+b_{16}\alpha_2\alpha_c+b_{17}\alpha_1\alpha_2+b_{18}\alpha_c^2+b_{19}\alpha_1^2+b_{20}\alpha_2^2
\end{aligned}
\tag{B34}
$$

Because α_c is arbitrary, given α_1 and α_2,

$$
b_9=b_6\alpha_1+b_7\alpha_2=b_8\alpha_1\alpha_2+b_{10}\alpha_1^2+b_{11}\alpha_2^2=0\,. \tag{B35}
$$

According to the similar logic with the α_1 and α_2, the followings hold.

$$
b_6=b_7=\cdots=b_{11}=0 \text{ and } b_{15}=b_{16}=\cdots=b_{20}=0\,. \tag{B36}
$$

Because coefficient of p is zero, at Equation (B32),

$$
\begin{aligned}
0 &= \left(e^{s_{11}}-1\right)\left(b_3\alpha_c+b_4\alpha_1+b_5\alpha_2\right)+\left(e^{s_2}-1\right)\left(b_{12}\alpha_c+b_{13}\alpha_1+b_{14}\alpha_2\right)\\
&+b_{21}\begin{pmatrix}(E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)v_c+f(s_{11})\alpha_c+\alpha_1 f_c(s_{11},s_{21})\\ +(E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)]-f_c(s_{11},s_{21})-\alpha_c)v_1\end{pmatrix}\\
&+b_{22}\begin{pmatrix}(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)v_c+f(s_{21})\alpha_c+\alpha_2 f_c(s_{11},s_{21})\\ +(E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)]-f_c(s_{11},s_{21})-\alpha_c)v_2\end{pmatrix}\\
&+b_{23}\begin{pmatrix}(E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)v_2+f(s_{21})\alpha_1+f(s_{11})\alpha_2\\ +(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)v_1\end{pmatrix}\\
&+2b_{24}((E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)]-f_c(s_{11},s_{21})-\alpha_c)v_c+f_c(s_{11},s_{21})\alpha_c)\\
&+2b_{25}((E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)v_1+f(s_{11})\alpha_1)\\
&+2b_{26}((E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)v_2+f(s_{21})\alpha_2)\\
&-E[g^s(s_{11}+\eta_1,s_{21}+\eta_2)]+g^s(s_1,s_2)+E[g^s(\eta_1,\eta_2)]
\end{aligned}
\tag{B37}
$$

Because v_c is arbitrary, coefficient of v_c is zero.

$$
\begin{aligned}
&b_{21}(E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)+b_{22}(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)\\
&+2b_{24}(E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)]-f_c(s_{11},s_{21})-\alpha_c)=0
\end{aligned}
\tag{B38}
$$

Now consider a random variable η_3 with $\eta_3 \overset{d}{\sim} \eta_2$ and $E[f_c(\eta_1,\eta_3)] \neq E[f_c(\eta_1,\eta_2)]$. Then,

$$
\begin{aligned}
&b_{21}(E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)+b_{22}(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)\\
&+2b_{24}(E[f_c(s_{11}+\eta_1,s_{21}+\eta_3)]-f_c(s_{11},s_{21})-E[f_c(\eta_1,\eta_3)])=0
\end{aligned}
\tag{B39}
$$

By subtracting these two equations, one can see that $b_{24}=0$ or

$$
E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)]=E[f_c(s_{11}+\eta_1,s_{21}+\eta_3)]+E[f_c(\eta_1,\eta_2)]-E[f_c(\eta_1,\eta_3)] \tag{B40}
$$

When we substitute

$$
(\eta_1,\eta_2)=\begin{cases}(\log(1+\sqrt{k}),\log(1+\sqrt{k})) & \Pr=1/2\\ (\log(1-\sqrt{k}),\log(1-\sqrt{k})) & \Pr=1/2\end{cases},
$$

$$
(\eta_1,\eta_3)=\begin{cases}(\log(1+\sqrt{k}),\log(1-\sqrt{k})) & \Pr=1/2\\ (\log(1-\sqrt{k}),\log(1+\sqrt{k})) & \Pr=1/2\end{cases}
$$

into Equation (B40) and multiply $2/(\ln(1+\sqrt{k})-\ln(1-\sqrt{k}))$ to the both hand side of the equation, and take the limit with $k \to 0$, we get

$$f_{c12}(s_{11}, s_{21}) = 1 \tag{B41}$$

Hence

$$f_c(s_{11}, s_{21}) = s_{11}s_{21} + F_1(s_{11}) + F_2(s_{21}) \tag{B42}$$

for some function F_1 and F_2. In addition the condition of $\lim_{s_{11}, s_{21} \to 0,0} \frac{f_c(s_{11}, s_{21})}{s_{11}s_{21}} = 1$ provides $f_c(s_{11}, s_{21}) = s_{11}s_{21}$. Therefore, (B40) implies $f_c(s_{11}, s_{21}) = s_{11}s_{21}$. When one substitute function f_c into previous of previous equation (B42), we get the following equation:

$$\begin{aligned} &b_{21}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) + b_{22}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) \\ &+ 2b_{24}(s_{11}E[\eta_2] + s_{21}E[\eta_1]) = 0 \end{aligned} \tag{B43}$$

When $s_{11} = 0$, (B43) is changed to $b_{22}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) + 2b_{24}s_{21}E[\eta_1] = 0$. Because η_1 can be chosen independently on s_{21} and η_2,

$$b_{24} = 0\,. \tag{B44}$$

Instead of Equation (B38), let us consider the coefficients of v_1 and v_2. Then adopting same logic from (B37) to (B44) yields

$$b_{21} = b_{22} = 0\,. \tag{B45}$$

Because the coefficient of v_1 is at Equation (B37), the equations (B44) and (B45) implies:

$$b_{23}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) + 2b_{25}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) = 0 \tag{B46}$$

Substituting $s_{11} = 0$ or $s_{21} = 0$ into the (B46) yields:

$$b_{23}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) = b_{25}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) = 0 \tag{B47}$$

Similarly, we can get an alternative form of (B47) by using the coefficient of v_2. The combination between these two yields the following

$$(E[f(s+\eta)]-f(s)-E[f(\eta)])=0 \text{ or } b_{23}=b_{25}=b_{26}=0. \tag{B48}$$

Here, $E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1=0$ is equivalent to

$$f(x)=2(e^x-1-x) \tag{B49}$$

by (Neuberger 2012). In sum, (B37) with (B44), (B45), and (B48) yields

$$\begin{aligned}0&=\left(b_3\alpha_c+b_4\alpha_1+b_5\alpha_2\right)\left(e^{s_{11}}-1\right)+\left(b_{12}\alpha_c+b_{13}\alpha_1+b_{14}\alpha_2\right)\left(e^{s_2}-1\right)\\&\quad+b_{23}\left(f(s_{21})\alpha_1+f(s_{11})\alpha_2\right)+2b_{25}f(s_{11})\alpha_1+2b_{26}f(s_{21})\alpha_2\\&\quad-E[g^s(s_{11}+\eta_1,s_{21}+\eta_2)]+g^s(s_{11},s_{21})+E[g^s(\eta_1,\eta_2)]\end{aligned} \tag{B50}$$

Substituting $\eta_1=\begin{cases}\log(1+\sqrt{k}) & \Pr=1/2\\ \log(1-\sqrt{k}) & \Pr=1/2\end{cases}$ and $\eta_2=0$ into the (B50) and taking limit yields:

$$\begin{aligned}0&=2b_4\left(e^{s_{11}}-1\right)+2b_{13}\left(e^{s_{21}}-1\right)+4b_{23}(e^{s_{21}}-1-s_{21})+8b_{25}(e^{s_{11}}-1-s_{11})\\&\quad-g^s_{11}(s_{11},s_{21})+g^s_1(s_{11},s_{21})+g^s_{11}(0,0)-g^s_1(0,0)\end{aligned} \tag{B51}$$

Similarly, when use $\eta_2=\begin{cases}\log(1+\sqrt{k}) & \Pr=1/2\\ \log(1-\sqrt{k}) & \Pr=1/2\end{cases}$ and $\eta_1=0$, we get the following relation

$$\begin{aligned}0&=2b_5\left(e^{s_{11}}-1\right)+2b_{14}\left(e^{s_{21}}-1\right)+4b_{23}(e^{s_{11}}-1-s_{11})+8b_{26}(e^{s_{21}}-1-s_{21})\\&\quad-g^s_{22}(s_{11},s_{21})+g^s_2(s_{11},s_{21})+g^s_{22}(0,0)-g^s_2(0,0)\end{aligned} \tag{B52}$$

Now let us consider η_1 and η_2 that are dependent each other. If we substitute (B53) or (B54) into Equation (B50) and subtract each other, then then taking limits yields the (B55).

$$(\eta_1,\eta_2)=\begin{cases}(\log(1+\sqrt{k}),\log(1+\sqrt{k})) & \Pr=1/2\\ (\log(1-\sqrt{k}),\log(1-\sqrt{k})) & \Pr=1/2\end{cases} \tag{B53}$$

$$(\eta_1,\eta_2)=\begin{cases}(\log(1+\sqrt{k}),\log(1-\sqrt{k})) & \Pr=1/2\\ (\log(1-\sqrt{k}),\log(1+\sqrt{k})) & \Pr=1/2\end{cases} \tag{B54}$$

$$0 = b_3\left(e^{s_{11}} - 1\right) + b_{12}\left(e^{s_{21}} - 1\right) - g_{12}^s(s_{11}, s_{21}) + g_{12}^s(0,0) \tag{B55}$$

Then the solutions of the (B51), (B52), and (B55) are given as

$$g^s(x,y) = b_3(e^x - x)y + b_{12}(e^y - y)x + h_1(x) + h_2(y) + b_{27}xy \tag{B56}$$

$$\begin{aligned} g^s(x,y) &= 2b_4(e^x x - e^x + x) - 2b_{13}(e^y - 1)x - 4b_{23}x(e^y - y - 1) \\ &+ 4b_{25}(2e^x x - 2e^x + x^2 + 4x) + e^x h_3(y) + h_4(y) + b_{28}x \end{aligned} \tag{B57}$$

$$\begin{aligned} g^s(x,y) &= 2b_{14}(e^y y - e^y + y) - 2b_5(e^x - 1)y - 4b_{23}y(e^x - x - 1) \\ &+ 4b_{26}(2e^y y - 2e^y + y^2 + 4y) + e^y h_5(x) + h_6(x) + b_{29}y \end{aligned} \tag{B58}$$

for some function $h_i(.)$ and constants b_{27}, b_{28}, and b_{29}. Therefore $g^s(x,y)$ is a linear combination of $e^x y, e^y x, xy, e^x x, e^x, x^2, x, e^y y, e^y, y^2, y$ and 1. Consistency about coefficients of $e^x y$ and $e^y x$ requires $b_5 = -\frac{1}{2}b_3 - 2b_{23}$ and $b_{13} = -\frac{1}{2}b_{12} - 2b_{23}$. Because $g^s(0,0)$ is zero, g and g^s are given by

$$\begin{aligned} g\left(s_1, s_2, V_1, V_2, V_c\right) &= b_0 V_c + b_1 V_1 + b_2 V_2 + \left(b_3 V_c + b_4 V_1 - \left(\frac{1}{2}b_3 + 2b_{23}\right)V_2\right)\left(e^{s_1} - 1\right) \\ &+ \left(b_{12}V_c - \left(\frac{1}{2}b_{12} + 2b_{23}\right)V_1 + b_{14}V_2\right)\left(e^{s_2} - 1\right) \\ &+ b_{23}\left(2\left(e^{s_2} - s_2 - 1\right)V_1 + V_1 V_2 + 2\left(e^{s_1} - s_1 - 1\right)V_2\right) \\ &+ b_{25}\left(4\left(e^{s_1} - s_1 - 1\right) + V_1\right)V_1 + b_{26}\left(4\left(e^{s_2} - s_2 - 1\right) + V_2\right)V_2 + g^s(s_1, s_2) \end{aligned} \tag{B59}$$

$$\begin{aligned} g^s(x,y) &= d_1(e^x - 1) + d_2 x + d_3(e^y - 1) + d_4 y + 4b_{23}xy + 4b_{25}x^2 + 4b_{26}y^2 \\ &+ b_3 e^x y + b_{12}e^y x + (2b_4 + 8b_{25})e^x x + (2b_{14} + 8b_{26})e^y y \end{aligned} \tag{B60}$$

(B59) and (B60) are arranged as

$$\begin{aligned} g\left(s_1, s_2, V_1, V_2, V_c\right) &= d_1(e^{s_{11}} - 1) + d_2 s_{11} + d_3(e^{s_{21}} - 1) + d_4 s_{21} + d_5 V_1 + d_6 V_2 + d_7 V_c \\ &+ d_8(V_1 - 2s_1)^2 + d_9(V_2 - 2s_2)^2 + d_{10}(V_1 - 2s_1)(V_2 - 2s_2) \\ &+ d_{11}e^{s_1}\left(2V_c - V_2 + 2s_2\right) + d_{12}e^{s_2}\left(2V_c - V_1 + 2s_1\right) \\ &+ d_{13}e^{s_1}(V_1 + 2s_1) + d_{14}e^{s_2}(V_2 + 2s_2) \end{aligned} \tag{B61}$$

where $d_5 = b_1 - b_4 + \frac{1}{2} b_{12} - 4b_{25}$, $d_6 = b_2 + \frac{1}{2} b_3 - b_{14} - 4b_{26}$, $d_7 = b_0 - b_3 - b_{12}$, $d_8 = b_{25}$,

$d_9 = b_{26}$, $d_{10} = b_{23}$, $d_{11} = \frac{1}{2} b_3$, $d_{12} = \frac{1}{2} b_{12}$, $d_{13} = b_4 + 4b_{25}$, $d_{14} = b_{14} + 4b_{26}$.[11]

Substituting it into equation (20) yields the following:

$$
\begin{aligned}
0 &= 2d_8(-v_1 - 2s_{11} + \alpha_1)(\alpha_1 + 2E[\eta_1]) + 2d_9(-v_2 - 2s_{21} + \alpha_2)(\alpha_2 + 2E[\eta_2]) \\
&\quad + d_{10}((-v_1 - 2s_{11} + \alpha_1)(\alpha_2 + 2E[\eta_2]) + (-v_2 - 2s_{21} + \alpha_2)(\alpha_1 + 2E[\eta_1])) \\
&\quad + (e^{s_{11}} - 1)\left(d_{13}(\alpha_1 - E[2\eta_1 e^{\eta_1}]) + d_{11}(2\alpha_c - 2E[\eta_2 e^{\eta_1}] - \alpha_2)\right) \\
&\quad + (e^{s_{21}} - 1)\left(d_{14}(\alpha_2 - E[2\eta_2 e^{\eta_2}]) + d_{12}\left(2\alpha_c - 2E[\eta_1 e^{\eta_2}] - \alpha_1\right)\right)
\end{aligned} \tag{B62}
$$

Since coefficients of v_1 and v_2 are zero, $E[f(\eta)] = E[-2\eta]$ or $d_8 = d_9 = d_{10} = 0$. In addition, because s_{11}, s_{21} are arbitrary,

$$
\begin{aligned}
0 &= d_{13}(\alpha_1 - E[2\eta_1 e^{\eta_1}]) + d_{11}(2\alpha_c - 2E[\eta_2 e^{\eta_1}] - \alpha_2) \\
0 &= d_{14}(\alpha_2 - E[2\eta_2 e^{\eta_2}]) + d_{12}\left(2\alpha_c - 2E[\eta_1 e^{\eta_2}] - \alpha_1\right)
\end{aligned} \tag{B63}
$$

(1) If d_{11} is not zero, for some constants k_1 and k_2, the following holds

$$
f_c(\eta_1, \eta_2) = \eta_2 e^{\eta_1} + \frac{1}{2}\alpha_2 + \frac{d_{13}}{2d_{11}}(2\eta_1 e^{\eta_1} - \alpha_1) + k_1\left(e^{\eta_1} - 1\right) + k_2\left(e^{\eta_2} - 1\right)
$$

Then $\frac{d_{13}}{2d_{11}}(2\eta_1 e^{\eta_1} - \alpha_1) + k_1\left(e^{\eta_1} - 1\right) = 0$ and $\frac{1}{2}\alpha_2 + k_2\left(e^{\eta_2} - 1\right) = -\eta_2$ because $\frac{f_c(x, y)}{xy} \to 1$ as $x, y \to 0$. Accordingly, $k_2 = -1$ because $\frac{f(x)}{x^2} \to 1$ as $x, y \to 0$. And it implies $k_1 = d_{13} = 0$. Therefore, $f_c(\eta_1, \eta_2) = \eta_2(e^{\eta_1} - 1)$, $f(\eta) = 2\left(e^{\eta} - \eta - 1\right)$. Then $d_{12} = d_{14} = 0$.

---

[11] The ten coefficients, $b_0, \cdots, b_5, b_{12}, b_{14}, b_{23}, b_{25}$, and b_{26} are replaced with $d_5, \cdots, d_{14}$. More precisely, $(d_5, d_8, d_{12}, d_{13})$ replace $(b_1, b_4, b_{12}, b_{25})$. $(d_6, d_9, d_{11}, d_{14})$ replace $(b_2, b_3, b_{14}, b_{26})$. d_{10} replaces b_{23}. And d_7 replaces b_0 given b_3 and b_{12}.

(2) Similarly, if d_{12} is not zero, $f_c(\eta_1,\eta_2)=\eta_1(e^{\eta_2}-1)$, $f(\eta)=2\left(e^{\eta}-\eta-1\right)$ and $d_{11}=d_{13}=d_{14}=0$

(3) or $d_{11}=d_{12}=d_{13}=d_{14}=0$, $f(\eta)=2\left(e^{\eta}-\eta-1\right)$ with arbitrary function f_c.

(4) $d_8=d_9=d_{10}=d_{11}=d_{12}=0$, $f(\eta)=2(\eta e^{\eta}-e^{\eta}+1)$ with arbitrary function f_c.

(5) $d_8=d_9=d_{10}=d_{11}=d_{12}=d_{13}=d_{14}=0$ with arbitrary functions f and f_c.

■

Table 1. Higher order moment swaps

Type of swap	Receiving at T	Paying at $j \in \{1, \cdots, N\}$	Cost at 0
Third comoment	$(S_{1,T} - S_{1,0})^2 (S_{2,T} - S_{2,0})$	$(\Delta S_{1,j})^2 \Delta S_{2,j} + \Delta S_{2,j} \Delta M_{2,0,j} + 2\Delta S_{1,j} \Delta M_{1,1,j}$	0
Non-zero fourth moment	$(S_{1,T} - S_{1,0})^4$	$(\Delta S_{1,j})^4 + 6(\Delta S_{1,j})^2 \Delta M_{2,0,j} + 4\Delta S_{1,j} \Delta M_{3,0,j} + 3(\Delta M_{2,0,j})^2$	$3M_{2,0,0}^2$
Fourth moment	$(S_{1,T} - S_{1,0})^4 - 3(S_{1,T} - S_{1,0})^2 M_{2,0,0}$	$(\Delta S_{1,j})^4 + 6(\Delta S_{1,j})^2 \Delta M_{2,0,j} + 4\Delta S_{1,j} \Delta M_{3,0,j} + 3(\Delta M_{2,0,j})^2$	0
Non-zero asymmetric fourth comoment	$(S_{1,T} - S_{1,0})^3 (S_{2,T} - S_{2,0})$	$(\Delta S_{1,j})^3 \Delta S_{2,j} + \Delta S_{2,j} \Delta M_{3,0,j} + 3\Delta M_{1,1,j} (\Delta S_{1,j})^2$ $+ 3\Delta M_{2,1,j} \Delta S_{1,j} + 3\Delta M_{2,0,j} \Delta S_{1,j} \Delta S_{2,j} + 3\Delta M_{2,0,j} \Delta M_{1,1,j}$	$3M_{2,0,0} M_{1,1,0}$
Asymmetric fourth comoment with *a*	$(S_{1,T} - S_{1,0})^3 (S_{2,T} - S_{2,0})$ $- 3a(S_{1,T} - S_{1,0})(S_{2,T} - S_{2,0}) M_{2,0,0}$ $- 3(1-a)(S_{1,T} - S_{1,0})^2 M_{1,1,0}$ with a constant *a*	$(\Delta S_{1,j})^3 \Delta S_{2,j} + \Delta S_{2,j} \Delta M_{3,0,j} + 3\Delta M_{1,1,j} (\Delta S_{1,j})^2$ $+ 3\Delta M_{2,1,j} \Delta S_{1,j} + 3\Delta M_{2,0,j} \Delta S_{1,j} \Delta S_{2,j} + 3\Delta M_{2,0,j} \Delta M_{1,1,j}$	0
Non-zero symmetric fourth comoment	$(S_{1,T} - S_{1,0})^2 (S_{2,T} - S_{2,0})^2$	$((\Delta S_{1,j})^2 + \Delta M_{2,0,j})((\Delta S_{2,j})^2 + \Delta M_{0,2,j}) + 2(\Delta M_{1,1,j})^2$ $+ 4\Delta M_{1,1,j} \Delta S_{1,j} \Delta S_{2,j} + 2\Delta M_{1,2,j} \Delta S_{1,j} + 2\Delta M_{2,1,j} \Delta S_{2,j}$	$M_{2,0,0} M_{0,2,0}$ $+ 2M_{1,1,0}^2$
Symmetric fourth comoment with *a*	$(S_{1,T} - S_{1,0})^2 (S_{2,T} - S_{2,0})^2$ $- 2(S_{1,T} - S_{1,0})(S_{2,T} - S_{2,0}) M_{1,1,0}$ $- a(S_{1,T} - S_{1,0})^2 M_{0,2,0}$ $- (1-a)(S_{2,T} - S_{2,0})^2 M_{2,0,0}$ with a constant *a*	$((\Delta S_{1,j})^2 + \Delta M_{2,0,j})((\Delta S_{2,j})^2 + \Delta M_{0,2,j}) + 2(\Delta M_{1,1,j})^2$ $+ 4\Delta M_{1,1,j} \Delta S_{1,j} \Delta S_{2,j} + 2\Delta M_{1,2,j} \Delta S_{1,j} + 2\Delta M_{2,1,j} \Delta S_{2,j}$	0

This table describes various (co)moment swaps. Each row represents structure of a swap. The second, the third, and the fourth column represent amount of receiving leg, paying leg, and initial cost respectively. Paying leg consists of the terms of realized cumulant and the receiving leg is a product of returns possibly with additional terms. Since each swap is constructed to be fair, some swaps require additional cost at time zero and they have a prefix, non-zero, at the name. The non-zero swaps are modified to be zero cost swaps by changing the receiving legs. Then expectation of receiving leg of a non-zero swap is (co)moment and expectation of receiving leg of a modified swap is (joint) cumulant. There are two kinds of comoment in the case of the fourth comoment. When the receiving leg is related to the product of square of returns, then it has affix, symmetric, at the name; otherwise, it has affix of asymmetric. In the case of the fourth comoments, there are various forms for zero cost swaps. We denote the variation with a constant *a*.

Table 2. Statistics of cumulants of the S&P 500 returns

This table represents descriptive statistics of cumulants, skewness, and kurtosis of S&P 500 returns from January 1996 to August 2014. Panel A shows the statistics of 30-calendar-day returns up to the last trading day of each option. The second column represents sample moments. The third and fourth column represent averages for the implied and realized values respectively. Implied cumulants are calculated through Equation (29). Realized cumulants are calculated through the expressions (7), (9), and (20). Numbers in parentheses are standard deviations for each term. The other panels are similar to the Panel A except the time horizon and frequency of sample.

A. 30 days

	Sample	Implied	Realized
2nd cumulant	0.24	0.40	0.32
(×100)		(0.37)	(0.50)
3rd cumulant	-0.11	-0.34	-0.22
(×1000)		(0.46)	(0.57)
Skewness	-0.90	-1.37	-1.11
		(0.50)	(0.72)
4th cumulant	0.18	0.80	0.25
(×10000)		(1.27)	(0.97)
Kurtosis	3.04	5.94	4.33
		(4.02)	(5.87)

B. 90 days

	Sample	Implied	Realized
2nd cumulant	0.75	1.18	0.89
(×100)		(0.80)	(0.97)
3rd cumulant	-0.11	-1.42	-0.92
(×1000)		(1.13)	(1.49)
Skewness	-0.16	-1.17	-1.11
		(0.38)	(0.49)
4th cumulant	0.62	3.76	1.28
(×10000)		(3.30)	(2.33)
Kurtosis	1.11	3.35	2.93
		(1.95)	(2.51)

C. 180 days

	Sample	Implied	Realized
2nd cumulant	1.65	2.32	1.93
(×100)		(1.36)	(2.03)
3rd cumulant	0.05	-3.01	-2.12
(×1000)		(2.09)	(2.44)
Skewness	0.02	-0.91	-1.03
		(0.33)	(0.53)
4th cumulant	3.04	6.75	3.84

(×10000)		(5.44)	(13.05)
Kurtosis	1.11	1.67	2.10
		(1.08)	(2.03)

D. 360 days

	Sample	Implied	Realized
2nd cumulant	3.97	4.63	7.81
(×100)		(2.27)	(28.59)
3rd cumulant	1.97	-4.23	6.40
(×1000)		(4.89)	(81.53)
Skewness	0.25	-0.50	-0.81
		(0.40)	(0.58)
4th cumulant	-12.91	8.63	468.85
(×10000)		(16.02)	(3652.54)
Kurtosis	-0.82	0.62	1.27
		(0.61)	(1.63)

Table 3. Adjusted skewness and kurtosis

Adjusted skewness represents monthly skewness which is calculated by skewness of *n*-month return; $\sqrt{n}$ times sample skewness of *n*-month return of the Table 2. In addition, adjusted kurtosis represents *n* times non-excess kurtosis of *n*-month return minus 3. When returns are i.i.d., each moment is proportional to the length of period, *n*. Accordingly, skewness is proportional to $1/\sqrt{n}$ and non-excess kurtosis is proportional to 1/*n* if the returns are i.i.d.. Likewise, adjust skewness or adjusted kurtosis should be irrelevant to the *n* if the returns are i.i.d..

Months (*n*)	1	3	6	12
Adjusted skewness	-0.90	-0.28	0.06	0.86
Adjusted kurtosis	3.04	9.34	21.68	23.19

Table 4. Time series regression of cumulants of S&P 500 returns

Panel A represents the time series regression about cumulants of monthly return of S&P 500 from January 1996 to August 2014. Each row represents the result of the regression with coefficients and t-values in parentheses. The first column represents the measure that we address. Within the measure, dependent variables are realized cumulants and the independent variables are implied or lagged realized cumulants. The other panels are similar to the Panel A except the time horizon.

A. 30 days

	Intercept	Implied	Realized(-1)	Adj. R2
2nd cum.	0.00	0.95		0.51
	(-1.23)	(6.31)		
	0.00		0.71	0.50
	(2.44)		(4.71)	
	0.00	0.54	0.37	0.55
	(-0.51)	(3.06)	(1.47)	
3rd cum.	0.00	0.82		0.43
	(0.98)	(4.00)		
	0.00		0.61	0.37
	(-2.02)		(2.46)	
	0.00	0.57	0.25	0.46
	(0.73)	(3.01)	(0.92)	
Skew	-0.20	0.67		0.22
	(-2.30)	(9.42)		
	-0.68		0.39	0.15
	(-7.59)		(4.68)	
	-0.20	0.52	0.19	0.24
	(-2.21)	(5.81)	(2.26)	
4th cum.	0.00	0.23		0.08
	(0.64)	(1.13)		
	0.00		-0.13	0.01
	(3.99)		(-0.46)	
	0.00	0.28	-0.24	0.13
	(0.83)	(1.62)	(-0.74)	
Kurt	1.48	0.48		0.10
	(3.25)	(6.27)		
	3.18		0.27	0.07
	(7.01)		(3.40)	
	1.38	0.38	0.16	0.12
	(3.01)	(4.94)	(2.31)	

B. 90 days

	Intercept	Implied	Realized(-1)	Adj. R2
2nd cum.	0.00	0.66		0.29
	(1.66)	(7.91)		
	0.00		0.48	0.22

	(4.66)		(3.85)	
	0.00	0.58	0.08	0.28
	(1.81)	(4.56)	(0.64)	
3rd cum.	0.00	0.56		0.17
	(-1.06)	(4.56)		
	0.00		0.35	0.11
	(-3.71)		(5.03)	
	0.00	0.48	0.08	0.16
	(-1.43)	(2.39)	(0.79)	
Skew	-0.17	0.80		0.39
	(-1.59)	(9.27)		
	-0.48		0.57	0.31
	(-4.69)		(6.49)	
	-0.17	0.59	0.22	0.40
	(-1.55)	(3.44)	(1.41)	
4th cum.	0.00	0.20		0.06
	(1.42)	(1.81)		
	0.00		0.32	0.09
	(2.93)		(2.15)	
	0.00	0.11	0.25	0.10
	(1.56)	(0.89)	(1.40)	
Kurt	0.39	0.75		0.34
	(1.01)	(6.81)		
	1.28		0.55	0.30
	(3.78)		(4.61)	
	0.26	0.51	0.32	0.40
	(0.76)	(3.53)	(2.74)	

C. 180 days

	Intercept	Implied	Realized(-1)	Adj. R2
2nd cum.	0.01	0.62		0.16
	(2.55)	(6.65)		
	0.01		0.27	0.06
	(5.18)		(2.52)	
	0.01	0.60	0.03	0.15
	(2.60)	(5.26)	(0.40)	
3rd cum.	0.00	0.53		0.19
	(-2.02)	(6.76)		
	0.00		0.29	0.07
	(-4.90)		(3.73)	
	0.00	0.63	-0.11	0.19
	(-1.77)	(4.49)	(-0.95)	
Skew	0.14	1.29		0.63
	(1.53)	(13.06)		
	-0.31		0.71	0.48

	(-3.45)		(9.44)	
	0.12	1.12	0.13	0.63
	(1.26)	(5.35)	(0.95)	
4th cum.	0.00	0.07		-0.01
	(0.89)	(0.18)		
	0.00		0.04	-0.01
	(2.14)		(0.47)	
	0.00	0.05	0.04	-0.03
	(0.87)	(0.14)	(0.48)	
Kurt	-0.30	1.43		0.57
	(-1.27)	(8.10)		
	0.87		0.60	0.32
	(3.18)		(4.13)	
	-0.30	1.25	0.15	0.57
	(-1.37)	(5.48)	(1.36)	

D. 360 days

	Intercept	Implied	Realized(-1)	Adj. R2
2nd cum.	0.02	1.18		-0.01
	(1.00)	(1.24)		
	0.08		-0.01	-0.02
	(2.19)		(-0.72)	
	0.02	1.27	-0.03	-0.02
	(0.90)	(1.21)	(-0.99)	
3rd cum.	0.01	0.47		-0.01
	(0.83)	(1.29)		
	0.01		0.00	-0.02
	(0.67)		(-0.31)	
	0.01	0.47	-0.01	-0.03
	(0.83)	(1.28)	(-0.45)	
Skew	-0.31	1.03		0.49
	(-3.66)	(9.11)		
	-0.39		0.58	0.32
	(-3.79)		(6.71)	
	-0.26	0.85	0.18	0.50
	(-2.87)	(5.16)	(1.47)	
4th cum.	0.08	-34.24		0.01
	(1.04)	(-0.91)		
	0.05		-0.02	-0.02
	(1.08)		(-1.12)	
	0.08	-36.47	-0.05	-0.01
	(1.03)	(-0.90)	(-1.00)	
Kurt	0.19	1.73		0.41
	(1.07)	(6.68)		
	0.87		0.37	0.10

	(3.81)		(3.08)	
	0.20	1.81	-0.05	0.40
	(1.11)	(5.90)	(-0.50)	

Table 5. Average of regression about comoments

Panel A represents average of time series regression results about comoments of monthly returns between S&P 500 and each stock contained in Dow Jones Industrial Average from January 1996 to August 2014. Each row represents the result of the regression with coefficients and t-values in parentheses. The first column represents the measure that we address. Within the measure, dependent variables are realized comoments and the independent variables are implied, lagged realized, and historical comoments; the implied and the realized moments are calculated as Section 3 describes and the historical comoments are calculated from the previous 24 monthly returns. Panel B is similar to the Panel A except that Panel B is about average of cross sectional regressions.

A. Average of time series regressions.

	Intercept	Implied	Realized(-1)	Historical	Adj. R2
covar	0.000	0.691			0.403
	(-1.304)	(11.510)			(13.477)
	0.001		0.495		0.322
	(15.013)		(11.042)		(11.394)
	0.002			0.568	0.037
	(5.899)			(4.992)	(3.641)
	0.000	0.521	0.134	0.010	0.429
	(2.246)	(7.532)	(3.247)	(0.057)	(13.846)
3rd comom	0.000	0.907			0.440
	(1.065)	(13.638)			(16.983)
	0.000		0.401		0.234
	(-18.145)		(9.255)		(10.566)
	0.000			0.676	0.053
	(-13.645)			(6.809)	(6.086)
	0.000	0.872	0.039	-0.184	0.466
	(1.212)	(13.435)	(1.155)	(-2.593)	(18.530)
beta	0.133	0.716			0.114
	(1.152)	(7.605)			(4.903)
	0.686		0.257		0.084
	(18.651)		(8.738)		(5.228)
	0.698			0.223	0.066
	(9.104)			(3.244)	(4.418)
	-0.022	0.586	0.111	0.150	0.167
	(-0.094)	(4.134)	(3.780)	(1.142)	(5.527)
gamma	0.310	0.648			0.045
	(2.092)	(5.382)			(2.966)
	0.952		0.042		0.005
	(26.977)		(1.835)		(0.520)
	0.992			-0.009	-0.004
	(26.816)			(-0.610)	(-0.688)
	0.339	0.644	-0.010	-0.027	0.035
	(2.225)	(4.913)	(-0.521)	(-1.167)	(2.141)

B. Average of cross sectional regressions.

	Intercept	Implied	Realized(-1)	Historical	Adj. R2
covar	0.000	0.668			0.282
	(-0.283)	(12.675)			(15.831)
	0.001		0.600		0.269
	(5.618)		(11.946)		(15.827)
	0.001			0.834	0.265
	(7.379)			(8.181)	(16.990)
	0.000	0.282	0.269	0.298	0.386
	(0.845)	(6.442)	(9.210)	(5.823)	(21.012)
3rd comom	0.000	0.617			0.358
	(-3.601)	(9.245)			(18.601)
	0.000		0.784		0.227
	(-6.649)		(3.331)		(12.863)
	0.000			0.107	0.075
	(-6.574)			(1.522)	(8.438)
	0.000	0.510	0.308	0.006	0.409
	(-3.713)	(10.603)	(2.942)	(0.203)	(21.448)
beta	0.036	0.861			0.282
	(0.748)	(18.166)			(15.831)
	0.455		0.508		0.269
	(20.007)		(21.210)		(15.827)
	0.485			0.459	0.265
	(23.011)			(22.536)	(16.990)
	0.109	0.374	0.240	0.212	0.386
	(2.539)	(8.198)	(12.797)	(12.072)	(21.012)
gamma	0.254	0.687			0.358
	(4.256)	(12.425)			(18.601)
	0.626		0.345		0.227
	(18.817)		(8.397)		(12.863)
	0.884			0.085	0.075
	(16.109)			(1.906)	(8.438)
	0.221	0.570	0.111	0.042	0.409
	(4.809)	(7.507)	(2.606)	(1.062)	(21.448)

Table 6. Return and realized moments of (co)moment portfolios

Panel A represents performance of portfolios that are constructed based on the rank of the implied volatility. We classify the firms in the DJIA index into the three groups, based on the model free implied variance at each month-end, with breakpoints 30% and 70%. Using the three groups, we make three equally weighted portfolios and zero cost portfolio which is denoted by 3-1. The numbers in the second column is the average of returns over the subsequent month. Similarly, the other columns present realized moments of return over the subsequent month. The last row represents t-value of the statistics for the 3-1 portfolio. Panel B and C are similarly constructed except that the portfolios are sorted based on the model free implied skewness or kurtosis at each month-end. The other Panels are similar except that the portfolios are sorted based on the realized moment of each month-end.

A. var_imp	return	volatility	beta	gamma	skewness	kurtosis
1 (lowest)	0.0060	0.0601	0.6696	0.7571	-0.2129	0.4505
2	0.0081	0.0764	0.9360	0.9829	-0.1533	0.3106
3 (highest)	0.0048	0.1045	1.2115	1.1614	-0.0187	0.3439
3-1	-0.0012	0.0444	0.5419	0.4042	0.1942	-0.1065
t(3-1)	-0.2795	15.7341	18.6191	9.9106	12.5295	-2.1609
B. skew_imp						
1	0.0031	0.0696	0.8378	0.9174	-0.2355	0.5461
2	0.0078	0.0772	0.9281	0.9559	-0.1415	0.3151
3	0.0082	0.0938	1.0535	1.0373	-0.0122	0.2428
3-1	0.0051	0.0242	0.2156	0.1199	0.2233	-0.3033
t(3-1)	1.4832	12.0778	9.1469	3.4865	15.5593	-5.6128
C. kurt_imp						
1	0.0056	0.0824	0.9939	1.0043	-0.1188	0.2503
2	0.0066	0.0776	0.9303	1.0021	-0.1376	0.3469
3	0.0071	0.0806	0.8946	0.8885	-0.1337	0.4957
3-1	0.0015	-0.0019	-0.0992	-0.1157	-0.0149	0.2454
t(3-1)	0.5322	-1.1838	-5.0005	-2.8297	-0.9675	4.7449
D. var_real(-1)						
1	0.0071	0.0631	0.7206	0.7985	-0.1972	0.4541
2	0.0075	0.0770	0.9240	0.9660	-0.1561	0.3687
3	0.0043	0.1005	1.1762	1.1415	-0.0311	0.2638
3-1	-0.0028	0.0374	0.4556	0.3430	0.1661	-0.1903
t(3-1)	-0.6844	13.3402	15.9114	5.6637	10.7673	-4.3224
E. β_real(-1)						
1	0.0072	0.0679	0.6813	0.7999	-0.1563	0.4433
2	0.0060	0.0756	0.9085	0.9738	-0.1476	0.3998
3	0.0063	0.0976	1.2352	1.1297	-0.0833	0.2319
3-1	-0.0008	0.0297	0.5539	0.3298	0.0730	-0.2114
t(3-1)	-0.2000	10.2679	19.7464	4.8118	4.6119	-5.5192
F. γ_real(-1)						
1	0.0068	0.0709	0.7527	0.8522	-0.1470	0.4466
2	0.0076	0.0758	0.9117	0.9583	-0.1524	0.3477
3	0.0046	0.0943	1.1598	1.0976	-0.0864	0.2992
3-1	-0.0022	0.0233	0.4072	0.2454	0.0605	-0.1475
t(3-1)	-0.6941	9.3553	15.1543	6.1452	3.9138	-3.4842
G. skew_real(-1)						

1	0.0081	0.0747	0.8937	0.9548	-0.2189	0.4011
2	0.0071	0.0783	0.9419	0.9596	-0.1430	0.3038
3	0.0038	0.0872	0.9796	0.9939	-0.0264	0.4030
3-1	-0.0043	0.0125	0.0859	0.0391	0.1924	0.0019
t(3-1)	-1.4363	5.0359	3.3964	1.3867	12.1968	0.0411
H. kurt_real(-1)						
1	0.0048	0.0811	0.9763	0.9747	-0.1265	0.2200
2	0.0061	0.0799	0.9569	0.9720	-0.1219	0.3555
3	0.0086	0.0787	0.8775	0.9580	-0.1467	0.5152
3-1	0.0038	-0.0024	-0.0988	-0.0167	-0.0202	0.2953
t(3-1)	1.5108	-1.9058	-5.2212	-0.4412	-1.3980	6.8189

Table 7. Fama and French 3 factor risk adjusted return

This table constructs the portfolios as the Table 6 describes. And then this table shows coefficients and t-values about time series regression of excess return of each portfolio on the Fama and French 3 factors; mkt, smb, and hml are market excess return, SMB factor, and HML factor respectively.

A. var_imp	Intercept	MKT	SMB	HML	Adj. R2
1	0.002	0.632	-0.291	0.125	0.603
	(1.332)	(14.561)	(-3.266)	(2.047)	
2	0.002	0.974	-0.262	0.195	0.802
	(1.164)	(22.550)	(-4.078)	(3.317)	
3	-0.006	1.457	0.077	0.536	0.782
	(-2.417)	(22.227)	(0.886)	(5.559)	
3-1	-0.008	0.825	0.368	0.411	0.457
	(-2.626)	(9.949)	(2.710)	(3.367)	
B. skew_imp					
1	-0.002	0.832	-0.259	0.131	0.695
	(-1.283)	(19.125)	(-4.225)	(2.028)	
2	0.002	0.957	-0.222	0.190	0.785
	(1.028)	(23.678)	(-3.212)	(3.141)	
3	-0.001	1.279	-0.005	0.531	0.733
	(-0.561)	(18.955)	(-0.056)	(5.018)	
3-1	0.001	0.448	0.254	0.400	0.213
	(0.273)	(5.234)	(2.176)	(2.968)	
C. kurt_imp					
1	-0.002	1.086	-0.209	0.277	0.779
	(-1.025)	(23.439)	(-2.683)	(3.880)	
2	0.000	0.934	-0.163	0.207	0.802
	(0.300)	(24.613)	(-2.787)	(4.051)	
3	0.000	1.057	-0.137	0.366	0.669
	(-0.185)	(16.557)	(-1.657)	(4.089)	
3-1	0.001	-0.029	0.072	0.090	-0.007
	(0.471)	(-0.370)	(0.626)	(0.818)	
D. var_real(-1)					
1	0.003	0.708	-0.298	0.156	0.636
	(1.566)	(14.523)	(-3.319)	(2.266)	
2	0.001	0.970	-0.291	0.173	0.840
	(1.051)	(31.984)	(-7.558)	(3.326)	
3	-0.006	1.387	0.114	0.541	0.741
	(-2.294)	(19.808)	(1.169)	(4.794)	
3-1	-0.009	0.678	0.411	0.385	0.358
	(-2.663)	(7.396)	(3.083)	(2.693)	
E. β_real(-1)					
1	0.003	0.680	-0.224	0.126	0.572
	(1.637)	(13.065)	(-2.511)	(1.863)	

2	-0.001	0.965	-0.195	0.297	0.769
	(-0.418)	(23.806)	(-2.675)	(4.698)	
3	-0.004	1.423	-0.081	0.397	0.762
	(-1.406)	(18.277)	(-0.920)	(3.476)	
3-1	-0.006	0.744	0.143	0.271	0.334
	(-1.897)	(6.974)	(1.001)	(1.809)	
F. γ_real(-1)					
1	0.001	0.853	-0.224	0.144	0.679
	(0.778)	(15.463)	(-2.805)	(2.392)	
2	0.001	0.990	-0.216	0.275	0.799
	(0.593)	(25.051)	(-3.492)	(4.082)	
3	-0.004	1.214	-0.057	0.416	0.723
	(-1.735)	(19.612)	(-0.636)	(3.870)	
3-1	-0.006	0.360	0.167	0.271	0.140
	(-1.792)	(4.106)	(1.297)	(1.988)	
G. skew_real(-1)					
1	0.003	0.836	-0.255	0.117	0.673
	(1.664)	(17.148)	(-3.579)	(1.607)	
2	0.000	0.995	-0.188	0.309	0.804
	(0.136)	(27.431)	(-3.413)	(5.855)	
3	-0.005	1.227	-0.064	0.396	0.799
	(-2.608)	(19.122)	(-0.797)	(4.414)	
3-1	-0.008	0.390	0.190	0.279	0.199
	(-3.013)	(4.351)	(1.838)	(2.385)	
H. kurt_real(-1)					
1	-0.002	1.017	-0.054	0.346	0.785
	(-1.555)	(25.606)	(-0.732)	(6.021)	
2	-0.001	1.044	-0.182	0.257	0.798
	(-0.609)	(26.825)	(-3.018)	(4.785)	
3	0.002	0.980	-0.269	0.233	0.704
	(1.134)	(17.813)	(-3.916)	(2.719)	
3-1	0.005	-0.037	-0.215	-0.113	0.026
	(1.935)	(-0.580)	(-2.920)	(-1.184)	

Table A1. Elements of a spanning set of functions that satisfy Equation (A5) for each pair (k_1,k_2)

k_1+k_2	(k_1,k_2)	Elements of a spanning set of functions for each pair (k_1,k_2)
1	(1,0)	s_1
2	(2,0)	s_1^2, $M_{2,0}$
	(1,1)	s_1s_2, $M_{1,1}$
3	(3,0)	s_1^3, $s_1M_{2,0}$, $M_{3,0}$
	(2,1)	$s_1^2s_2$, $s_1M_{1,1}$, $s_2M_{2,0}$, $M_{2,1}$
4	(4,0)	s_1^4, $s_1M_{3,0}$, $\frac{1}{2}M_{2,0}^2+M_{2,0}s_1^2$
	(3,1)	$s_1^3s_2$, $M_{2,0}M_{1,1}+M_{1,1}s_1^2+M_{2,0}s_1s_2$, $M_{2,1}s_1$, $M_{3,0}s_2$
	(2,2)	$s_1^2s_2^2$, $M_{2,0}M_{0,2}+M_{0,2}s_1^2+M_{2,0}s_2^2$, $\frac{1}{2}M_{1,1}^2+M_{1,1}s_1s_2$, $M_{1,2}s_1$, $M_{2,1}s_2$
5	(5,0)	s_1^5, $M_{2,0}M_{3,0}+\frac{3}{2}M_{2,0}^2s_1+M_{3,0}s_1^2+M_{2,0}s_1^3$
	(4,1)	$s_1^4s_2$, $4M_{3,0}M_{1,1}+4M_{1,1}s_1^3+4M_{3,0}s_1s_2+12M_{2,0}M_{1,1}s_1+3M_{2,0}^2s_2+6M_{2,0}M_{2,1}+6M_{2,1}s_1^2+6M_{2,0}s_1^2s_2$
	(3,2)	$s_1^3s_2^2$, $M_{3,0}M_{0,2}+6M_{2,1}M_{1,1}+3M_{1,2}M_{2,0}+3M_{2,0}M_{0,2}s_1+6M_{1,1}^2s_1+6M_{2,0}M_{1,1}s_2+M_{3,0}s_2^2+6M_{2,1}s_1s_2+3M_{1,2}s_1^2+M_{0,2}s_1^3$ $+6M_{1,1}s_1^2s_2+3M_{2,0}s_1s_2^2$
6	(6,0)	s_1^6, $\frac{1}{2}M_{3,0}^2+3M_{3,0}M_{2,0}s_1+M_{3,0}s_1^3$
	(5,1)	$s_1^5s_2$, $M_{3,0}M_{2,1}+2M_{3,0}M_{1,1}s_1+3M_{2,0}M_{2,1}s_1+M_{3,0}M_{2,0}s_2+M_{2,1}s_1^3+M_{3,0}s_1^2s_2$
	(4,2)	$s_1^4s_2^2$, $M_{3,0}M_{1,2}+M_{3,0}M_{0,2}s_1+3M_{1,2}M_{2,0}s_1+2M_{3,0}M_{1,1}s_2+M_{1,2}s_1^3+M_{3,0}s_1s_2^2$, $\frac{1}{2}M_{2,1}^2+2M_{2,1}M_{1,1}s_1+M_{2,1}M_{2,0}s_2+M_{2,1}s_1^2s_2$

(3,3)	$s_1^3 s_2^3, M_{3,0}M_{0,3} + 3M_{2,0}M_{0,3}s_1 + 3M_{3,0}M_{0,2}s_2 + M_{0,3}s_1^3 + M_{3,0}s_2^3, M_{2,1}M_{1,2} + 2M_{1,1}M_{1,2}s_1 + M_{0,2}M_{2,1}s_1 + 2M_{2,1}M_{1,1}s_2 + M_{1,2}M_{2,0}s_2 + M_{1,2}s_1^2 s_2 + M_{2,1}s_1 s_2^2$

Each row of this table represents elements of spanning set of functions that satisfies Equation (A5) and $g(l_1S_1, l_2S_2, M(l_1S_1, l_2S_2)) = l_1^{k_1} l_2^{k_2} g(S_1, S_2, M(S_1, S_2))$ for each pair (k_1,k_2) with $k_1 \geq k_2$. The cases of $k_2 > k_1$ are omitted because they are represented by symmetry.

Table A2. Elements of basis of functions that satisfies Equation (A5) for each pair (k_1,k_2)

k_1+k_2	(k_1,k_2)	Elements of basis for each pair (k_1,k_2)
1	(1,0)	s_1
2	(2,0)	s_1^2, $M_{2,0}$
	(1,1)	$s_1 s_2$, $M_{1,1}$
3	(3,0)	$s_1^3 + 3s_1 M_{2,0}$, $M_{3,0}$
	(2,1)	$s_1^2 s_2 + 2s_1 M_{1,1} + s_2 M_{2,0}$, $M_{2,1}$
4	(4,0)	$s_1^4 + 6s_1^2 M_{2,0} + 4s_1 M_{3,0} + 3M_{2,0}^2$
	(3,1)	$s_1^3 s_2 + s_2 M_{3,0} + 3(M_{2,0} M_{1,1} + M_{1,1} s_1^2 + M_{2,0} s_1 s_2 + M_{2,1} s_1)$
	(2,2)	$s_1^2 s_2^2 + M_{2,0} M_{0,2} + M_{0,2} s_1^2 + M_{2,0} s_2^2 + 2M_{1,1}^2 + 4M_{1,1} s_1 s_2 + 2M_{1,2} s_1 + 2M_{2,1} s_2$
5	(5,0)	N/A
	(4,1)	N/A
	(3,2)	N/A
6	(6,0)	N/A
	(5,1)	N/A
	(4,2)	N/A
	(3,3)	N/A

Each row of this table represents elements of basis of functions that satisfies the Aggregation Property and $g(l_1 S_1, l_2 S_2, M(l_1 S_1, l_2 S_2)) = l_1^{k_1} l_2^{k_2} g(S_1, S_2, M(S_1, S_2))$ for each pair (k_1,k_2).

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