Bayesian Approach for Identifying Contagion

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ABSTRACT

We propose a Bayesian approach to identify the excessive comovement of two markets as a contagion. This goal is technically achieved by linking a latent factor model and single equation error correction model and testing the breaks in the short-term and long-term relationships and correlatedness in the linked model. We find that a short-term relationship representing a systematic volatility ratio between two markets plays a key role in contagion dynamics. When long-term relationship or correlatedness is broken, the cause is determined by calculating posterior probabilities. If the cause is a break in the short-term relationship, a contagion is formally declared.

Keywords: Bayesian approach, contagion test, market integration, volatility spillover

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I. Introduction

In the existing literature, it is widely accepted that disentangling contagion from interdependence is critical for correctly identifying contagion during financial crises. Difficulties in identifying contagion lie in the fact that a contagion is not simply revealed by an increased correlation of performance indicators during a crisis period because increasing correlation coefficients are polluted by the rising volatility conditions, which are almost invariably associated with periods of financial stress. Most contagion tests define a contagion as “a correlation in excess of that expected via fundamentals” as noted by Bekaert, Harvey, and Ng (2005). Later, Bekaert, Ehtmann, and Fratzscher (2014) more directly related contagion as "the comovement in excess of that implied by the factor model". Bekaert et al. (2005) acknowledges the problem of identifying both the relevant fundamentals and how they are linked to asset correlation. To resolve these issues, factor decomposition models are introduced based on the fact that the volatility of the target market may change both for reasons associated with the common factor and its own market-specific factor. Thus, contagion tests have been refined continuously over the years as the sources of correlation biases are identified by removing confounding effects[1]. There are two critical issues (Dungey and Renault (2018)) when identifying correlation bias sources: (i) how asset correlation is related to common and market-specific factors during tranquil and crisis periods (Forbes and Rigobon (2002), Bekaert et al. (2005), Corsetti, Pericoli, and Sbracia (2005)) and (ii) how all time-varying components of return volatilities are related to structural breaks in crisis times (Rodriguez (2007), Bussetti and Harvey (2011), Bae, Karolyi, and Stulz (2003), Baur and Schulze (2005), Boyson, Stahel, and Stulz (2010), Aït-Sahalia, Cacho-Diaz, and Laeven (2015)).

The originality of our approach is to apply the Bayesian method to detect the comovement in excess of that implied by the established model as a contagion. Simply speaking, when correlated relations between target and source markets are broken in terms of long-term relationship or correlatedness, its cause is determined by calculating posterior probabilities. If

[1] A situation in which the effect between the target and source market is distorted by the presence of other factors. This situation is typically resolved by introducing a proper factor model.

the cause is breaks in the underlying relationship between the two markets under integration, contagion is addressed formally. In this paper, this approach completely resolves two critical issues (i) and (ii) above (refers to Section II.D).

To incorporate the Bayesian approach, this paper links latent factor model (LFM) and single equation error correction model (SEECM). LFM assumes market-specific volatilities together with common systematic volatility for describing market return under market integration. SEECM describes the dynamic process of market returns to equilibrium from a state of disequilibrium due to a shock using the assumption that two market return time series exhibit an equilibrium linear relationship that determines both short- and long-term behavior (Banerjee, Galbraith, and Dolado (1990), Banerjee, Dolado, Galbraith, and Hendry (1993), Davidson, Hedry, Srba, and Yeo (1978)). By linking the two models via autoregressive AR(1) error (Proposition 1 below), it is found that short-term relationship between two markets is entirely embedded into a long-term relationship and correlatedness between two markets. Based on this linked model, we establish the null market integration hypothesis (MIH) for tranquil periods (refer to Proposition 2) and find that the short-term relationship between two markets (or 'common-factor loading ratio between two markets') is the main backbone of the linked model under the MIH (refer to equation (10) below). These findings make an accurate and concise definition of contagion possible; a contagion is defined as an occurrence of breaks in the long-term relationship or correlatedness between two markets mainly caused by breaks in the short-term relationship between two markets. This definition makes sense because a contagion would be in effect when the short-term effect of a shock is not controlled properly at the beginning of the crisis period and continues to cause breaks in long-term relationship or correlatedness between two markets.

We organize this paper into the following sections. Section II discusses market modeling under integration. Section III describes contagion dynamics and the Bayesian testing procedure. Section IV offers concluding remarks.

II. Market modeling under integration

A. Latent factor model (LFM) and single equation error correction model (SEECM)

Assume that there are two market stationary returns modeled as follows:

$$X_t = \theta_x W_t + \delta_x u_{x,t}\ , \qquad Y_t = \theta_y W_t + \delta_y u_{y,t} \tag{1}$$

where W_t represents a common factor with loadings θ_x and θ_y. The common factor W_t represents market fundamentals, and its loadings θ_x and θ_y indicate the systematic risk levels of markets X and Y, respectively. The terms $u_{x,t}$ and $u_{y,t}$ in equation (1) are idiosyncratic factors unique to markets X and Y with the loadings δ_x and δ_y, respectively. Factors W_t, $u_{x,t}$, and $u_{y,t}$ are assumed to follow stochastic processes with zero mean and unit variance as follows:

$$W_t \sim (0,1),\ \ u_{x,t} \sim (0,1),\ \ u_{y,t} \sim (0,1).$$

To complete the specification of the model, all factors are assumed to be independent:

$$\mathrm{E}(u_{x,t}u_{y,t}) = 0, \quad \mathrm{E}(u_{x,t}W_t) = 0, \quad \mathrm{E}(u_{y,t}W_t) = 0.$$

To highlight the interrelationships between the two market returns in equation (1), the variances and covariance are represented as follows:

$$\mathrm{Cov}(X_t, Y_t) = \theta_x\theta_y\ , \quad \mathrm{Var}(X_t) = \theta_x^2 + \delta_x^2, \qquad \mathrm{Var}(Y_t) = \theta_y^2 + \delta_y^2.$$

Note that the following equilibrium relationship between X_t and Y_t exists by assuming LFM equation (1):

$$Y_t = \frac{\theta_y}{\theta_x}X_t - \frac{\delta_x}{\theta_x}u_{x,t} + \delta_y u_{y,t}. \tag{2}$$

Assuming that X_t and Y_t are two stationary market returns, the SEECM is specified as

$$\Delta Y_t = \alpha + \beta_0\Delta X_t + \beta_1 Y_{t-1} + \beta_2 X_{t-1} + \varepsilon_t = \alpha + \beta_0\Delta X_t + \beta_1(Y_{t-1} - \gamma X_{t-1}) + \varepsilon_t \tag{3}$$

where $\gamma = -\frac{\beta_2}{\beta_1}$, $\Delta Y_t \equiv Y_t - Y_{t-1}$, $\Delta X_t \equiv X_t - X_{t-1}$, and ε_t is the independent and identically distributed (iid) error. The part of the equation within the parentheses in equation (3) is the error correction mechanism, where $(Y_{t-1} - \gamma X_{t-1}) = 0$ when X and Y are in

equilibrium. The coefficient β_0 *specifies the short-term effects* of an increase in X on an increase in Y, while β_1 describes the speed at which X and Y return to equilibrium from a state of disequilibrium. The coefficient γ *specifies the long-term effects* of X on Y. Note that when $-1 < \beta_1 < 0$ ($\beta_1 > 0$), the system converges to equilibrium (diverges from equilibrium). Since β_1 represents the speed of return to equilibrium (and is therefore the scaled inverse of market-specific volatility of market Y, refer to footnote 4 for its details) and $\beta_2 = -\gamma\beta_1$ is the long-term relationship adjusted by the corresponding volatility, β_2 measures the *correlatedness* between markets X and Y. Recall that the correlatedness often suffers from bias in testing contagion (Forbes and Rigobon (2002)[2]). Dynamic specifications by SEECM allow us to estimate and test for both short- and long-term effects that help us better understand contagion dynamics between two markets. When a shock hits a market, we expect an immediate short-term effect of the shock on the other market. If the shock effect continues, it tends to have a long-term effect and may cause deviation from the equilibrium (or breaks in long-term relationship or correlatedness) between two markets. Thus, SEECM equation (3) is suited to dynamically model how a shock occurring in one market influences the other market during crisis. Engle and Granger (1987) propose equation (3) as a two-step error correction model for two or more cointegrated time series, whereas the SEECM employed herein applies to two stationary time series (see De Boef and Keele (2008) for details). The concepts of error correction, short-term effects, and equilibrium are not unique to cointegrated data[3]. Note that when deviation from the long run equilibrium occurs, its recovery process, which involves dynamism between X and Y, is modeled by SEECM.

A simple example demonstrates the interpretation of coefficients in equation (3). Let's say we regress the first difference of given one market returns (ΔY_t) on one lag of the market returns (Y_{t-1}), one lag of the other market returns (X_{t-1}), and the first difference of the other market returns (ΔX_t) as noted in equation (3). The coefficients are $\beta_0 = 0.3$, $\beta_1 = -0.5$, and $\beta_2 = 1.0$, which implies the long-term effects of X on Y, $\gamma = 2$. If X market return were to

[2] Forbes and Rigobon (2002) note that cross-market correlations increase during a crisis period without breaks in the long-term relationship between markets.

[3] We assume that a long run equilibrium or relationship exists between stationary Y_t and stationary X_t during a tranquil period. Assuming stationary (X_t, Y_t) is technically sound in the sense that their relationship is invariant over time during a tranquil period.

increase five points ($\Delta X_t = 5$), market Y return will first increase 1.5 points immediately (5 * 0.3, the coefficient of β_0). Although this increase in X market return might disturb the equilibrium, the SEECM implies that market Y return and market X return have an equilibrium relationship through the error correction process, that is, a 10-point increase in Y (5 * 2, the coefficient of γ). However, the increase in market Y return (or re-equilibration) is not immediate, occurring over future time periods at a rate dictated by β_1. The largest portion of the movement in market Y return will occur in the next time period when a 5-point increase in Y (10 * 0.5, the coefficient of β_1) will occur. In the following time period, (t + 1), market Y return will increase 2.5 points, increasing 1.25 points at t + 2 and .63 points in t+3 and so on until market Y return has increased ten points. Thus, market X return has two effects on market Y return: one effect that occurs immediately and another effect that is dispersed across future time periods. This error correction process might be in trouble during crisis periods. For instance, an occurrence of $\beta_1 > 0$ or $\beta_1 < -1$ during crisis periods might lead to extreme divergence of market returns.

B. Linking LFM and SEECM

We link the SEECM equation (3) to the LFM equation (1) to model market dynamics between two markets under integration. This linking process yields a proper model for identifying contagion via a Bayesian approach. To link two models, we introduce autoregressive (AR) errors $u_{x,t}$ and $u_{y,t}$ for LFM as follows:

Proposition 1. *Suppose that we employ an AR(1) model for idiosyncratic factors* $u_{x,t}$ *and* $u_{y,t}$ *in equation (1),*

$$u_{x,t} = \rho_x u_{x,t-1} + a_{u,t} \qquad u_{y,t} = \rho_y u_{y,t-1} + b_{u,t} \tag{4}$$

where $E(a_{u,t}W_t) = 0,\ E(a_{u,t}u_{x,t}) = 0,\ E(b_{u,t}W_t) = 0,\ E(b_{u,t}u_{y,t}) = 0, E(a_{u,t}b_{u,t}) = 0$, $a_{u,t} \sim iid\ (0,1), and\ b_{u,t} \sim iid\ (0,1)$. *Then, we have linked the SEECM to the LFM as follows:*

$$\Delta Y_t = \frac{\theta_y}{\theta_x}\Delta X_t - (1-\rho_y)\left(Y_{t-1} - \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}X_{t-1}\right) + \varepsilon_{Lt} \tag{5}$$

where $\varepsilon_{Lt} = \delta_y b_{u,t} - \frac{\theta_y}{\theta_x}\delta_x a_{u,t} + (\rho_x - \rho_y)\theta_y W_{t-1}$ *are iid over* t, $0 < \rho_y < 1$, and $Var(\varepsilon_{Lt}) = \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2 + (\rho_x - \rho_y)^2 {\theta_y}^2$.

Proof of Proposition 1 is given in Appendix A.

Remark 1. Recalling that θ_x and θ_y indicate the systematic risk of markets X and Y, respectively, due to a common factor W_t in equation (1), $\frac{\theta_y}{\theta_x}$ refers to the ratio of systematic risk of market Y to that of market X. The variance of idiosyncratic factors of X and Y are $\mathrm{Var}(u_{x,t}) = 1/(1-\rho_x^2)$ and $\mathrm{Var}(u_{y,t}) = 1/(1-\rho_y^2)$ by equation (4), respectively. It is easy to expect that the market-specific volatilities become high as ρ_x and ρ_y are close to 1 during a crisis period. In this sense, $|\rho_x|(<1)$ and $|\rho_y|(<1)$ imply reasonable predictability of idiosyncratic factor of X and Y, respectively. Also the condition $0 < \rho_y < 1$ is necessary for the equilibrating mechanism to work. Proposition 1 implies that policy makers of X and Y have their own market control measures via ρ_x and ρ_y.

Remark 2. Using Proposition 1, we have the following:

$$\Delta Y_t = \frac{\theta_y}{\theta_x}\Delta X_t - (1-\rho_y)\left(Y_{t-1} - \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}X_{t-1}\right) + \varepsilon_{Lt}$$
$$= \beta_{L0}\Delta X_t + \beta_{L1}(Y_{t-1} - \gamma_L X_{t-1}) + \varepsilon_{1t} = \beta_{L0}\Delta X_t + \beta_{L1}Y_{t-1} + \beta_{L2}X_{t-1} + \varepsilon_{Lt}. \quad (6)$$

where $\beta_{L0} = \frac{\theta_y}{\theta_x}$, $\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ and $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ correspond to short-term effect, long-term effect and correlatedness for the linked model, respectively[4]. It should be mentioned here that based on equation (5) and equation (6), $\beta_{L0} = \frac{\theta_y}{\theta_x}$ is entirely embedded in γ_L, β_{L2} and ε_{Lt}. This feature not only plays a key role in defining a contagion formally but also provides critical information for implementing the Bayesian approach. Refer to Section 3.3. Note that ε_{Lt} of equation (5) depends on W_{t-1}, suggesting endogeneity, which usually leads

[4] Note that β_1 represents the scaled inverse of market-specific volatility of market Y from $\beta_{L1} = -(1-\rho_y) = \frac{-(1-\rho_y)(1+\rho_y)}{(1+\rho_y)} = \frac{1}{Var(u_{y,t})}\frac{-1}{(1+\rho_y)}$.

to causality direction runs in both direction (from *X* to *Y* and from *Y* to *X*). Refer to Dungey, Fry, Gonza´ lez-Hermosillo, and Martin (2005).

C. Market integration hypothesis

In this subsection, we establish a market integration hypothesis (MIH) that corresponds to a market condition during tranquil periods. Since Proposition 1 specifies general market dynamics between X and Y, one may base a purely tranquil period as null against crisis period. We start with the following Proposition 2.

Proposition 2. *Assume the null MIH as follows:*

$$\textit{MIH: } \rho_x = \rho_y = \rho_0 \ \textit{ for some } \ 0 < \rho_0 < 1, \quad \frac{\theta_y}{\theta_x} \neq 1, \textit{ and } \quad \theta_x \theta_y \neq 0. \tag{7}$$

Then, under the null MIH, equation (5) might be rewritten as follows:

$$\Delta Y_t = \frac{\theta_y}{\theta_x} \Delta X_t - (1 - \rho_0)\left(Y_{t-1} - \frac{\theta_y}{\theta_x} X_{t-1}\right) + \varepsilon_{Mt} \tag{8}$$

where $\varepsilon_{Mt} = \delta_y b_{u,t} - \frac{\theta_y}{\theta_x} \delta_x a_{u,t}$ *is an iid error with* $E(\varepsilon_{Mt}) = 0$ *and finite variance of* $Var(\varepsilon_{Mt}) = \delta_y^2 + \left(\frac{\theta_y}{\theta_x} \delta_x\right)^2$.

Proof of Proposition 2. Equation (8) follows easily by applying equation (7) to equation (5).

Remark 3. The null MIH above implies a situation of two markets that lies midway between the two extremes of perfect integration into one market and two completely independent markets. If two markets are completely integrated into one, then $\delta_x = \delta_y$, $a_{u,t} = b_{u,t}$, $\theta_x = \theta_y$, and $\rho_x = \rho_y = \rho_0$ for any $0 \leq \rho_0 < 1$. Then, $\Delta Y_t = \Delta X_t$. In contrast, if two markets are completely independent, then $\theta_y = 0$ or $\theta_x = 0$. When $\theta_y = 0$ $(\theta_x = 0)$, we have $\Delta Y_t = -(1 - \rho_y) Y_{t-1} + \delta_y b_{u,t}$ $(\Delta \mathrm{X_t} = -(1 - \rho_x) X_{t-1} + \delta_x a_{u,t})$, where ΔY_t $(\Delta \mathrm{X_t})$ is not affected by any factor related to X_t (Y_t). In other words, the markets do not have a common factor. Of course, $X_t = \delta_x u_{x,t}$ and $Y_t = \delta_y u_{y,t}$ when $\theta_x = \theta_y = 0$. Using Proposition 2, we have the following:

$$\Delta Y_t = \frac{\theta_y}{\theta_x}\Delta X_t - (1-\rho_0)\left(Y_{t-1} - \frac{\theta_y}{\theta_x}X_{t-1}\right) + \varepsilon_{Mt}$$

$$= \beta_{M0}\Delta X_t + \beta_{M1}(Y_{t-1} - \gamma_M\ X_{t-1}) + \varepsilon_{Mt}\ = \beta_{M0}\Delta X_t + \beta_{M1}Y_{t-1} + \beta_{M2}X_{t-1} + \varepsilon_{Mt} \quad (9)$$

where $\beta_{M0} = \gamma_M = \frac{\theta_y}{\theta_x}$ and $\beta_{M2} = \frac{\theta_y(1-\rho_0)}{\theta_x}$ correspond to short-term effect, long-term effect and correlatedness for the MIH, respectively. This equation subsequently implies the basic linear relationship between two markets during a tranquil period, that is,

$$\Delta Y_t := \frac{\theta_y}{\theta_x}\Delta X_t \quad (10)$$

Remark 4. The MIH implies that the market-specific volatility is the same in markets X and Y ($\rho_x = \rho_y = \rho_0$), and they share a common factor $(\theta_x\theta_y \neq 0)$ with different levels of systematic risk ($\frac{\theta_y}{\theta_x} \neq 1$). This notion is consistent with financial market theory. The theory indicates that only global systematic risks are priced in two fully integrated markets or the integration leads to the conversion of a market-specific risk of each market into a common risk, that is, the local market-specific risks are fully diversified for integrated markets (Lehkonen (2015)). In other words, by letting $\rho_x = \rho_y$ in equation (4), the MIH achieves the two fully integrated markets without any endogeneity problem[5]. The condition $0 < \rho_x = \rho_y = \rho_0 < 1$ is essential for the convergence to the equilibrium in equation (9) (i.e., $-1 < \beta_{M1} < 0$ in equation (9))[6]. Hence, it is reasonable to discuss market contagion by testing the MIH as a proper null hypothesis. Testing MIH with $\Delta Y_t := \frac{\theta_y}{\theta_x}\Delta X_t$ of equation (10) enables us to accurately detect the comovement in excess of that implied by the linked model.

[5] No endogeneity under the MIH does not mean that the contemporaneous causality direction runs only in one direction, from X to Y. It means that two markets are completely integrated or completely exchangeable via equation (8).

[6] Note that an immediate endogeneity problem exists; hence, the contemporaneous causality direction runs in both directions when $0 < \rho_x \neq \rho_y < 1$.

D. Resolving the two critical issues

In this subsection, we show that the linked model resolves the two critical issues for identifying contagions discussed in the Introduction: (i) how asset correlation is related to common and market-specific factors during tranquil and crisis periods and (ii) how all time-varying components of return volatilities are related to structural breaks in crisis times. For the first issue, note from equation (6) that long-term effect ($\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$) and correlatedness ($\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$) are related to a common factor via $\frac{\theta_y}{\theta_x}$ and to market specific-factors through $(1-\rho_x)$ and $(1-\rho_y)$ under general situations, including crisis periods. Under MIH (or during a tranquil period), these factors are reduced to $\gamma_M = \frac{\theta_y}{\theta_x}$ and $\beta_{M2} = \frac{\theta_y(1-\rho_0)}{\theta_x}$ in equation (9). Thus, our linked model shows how correlation between two markets is related to common and market-specific factors during tranquil and crisis periods. For the second issue, note that our linked model defines the volatility spillover clearly and simply by taking into account systematic (common) as well as market-specific volatility spillover. Recalling that $\frac{\theta_y}{\theta_x}$ is related to systematic volatility and that ρ_x and ρ_y are related to market-specific volatility, the volatility spillover occurs due to changes in ($\frac{\theta_y}{\theta_x}$, ρ_x, ρ_y) caused by a shock. These volatility changes cause direct changes in short-term effect ($\beta_{L0} = \frac{\theta_y}{\theta_x}$), long-term effect ($\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$), correlatedness ($\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$) or convergence speed to equilibrium ($\beta_{L1} = -(1-\rho_y)$). When MIH is rejected or ($\frac{\theta_y}{\theta_x}$, ρ_0, ρ_0) under MIH changes to ($\frac{\theta'_y}{\theta'_x}$, ρ'_x, ρ'_y), error volatility under MIH changes from $\mathrm{Var}(\varepsilon_{Mt}) = \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2$ to

$$\mathrm{Var}(\varepsilon_{Lt}) = \delta_y^2 + \left(\frac{\theta'_y}{\theta'_x}\delta_x\right)^2 + (\rho'_x - \rho'_y)^2 \theta'^2_y \tag{11}$$

Thus, equation (11) defines the volatility spillover related to systematic ($\frac{\theta'_y}{\theta'_x}$) as well as market-specific volatility (ρ'_x, ρ'_y). This explains how all time-varying components of return volatilities are related to structural breaks in crisis times.

In the literature, volatility spillover is often treated by tail dependence of market return. Indeed, Rodriguez (2007) notes that structural breaks in tail dependence are an actual dimension of the contagion effects and tests them by applying copulas. Bussetti and Harvey (2011) use time-varying copulas to test financial contagion via tail events. Using coexceedance measures or quantiles, a number of studies also detect the contagion via tail events (Bae et al. (2003), Baur and Schulze (2005), Boyson et al. (2010)). Aït-Sahalia et al. (2015) measure contagion effects via extreme tail event of mutually exciting jumps. Recently, Dungey and Renault (2108) apply a GARCH common feature approach for identifying contagion.

III. Contagion dynamics and test procedure

A. Defining contagion and testing breaks in the relationship between markets

In this section, we define contagion using the concept of a "*break*" in the relationship between two markets. A shock that occurs in a market might cause a break in the established relationship with the other market and result in a contagion. We consider three types of breaks in market relationships: a break in short-term relationship or short-term break (SB), a break in long-term relationship or a long-term break (LB) and a break in correlatedness or a correlatedness break (CRB). A contagion is expected to occur when a shock causes a break in an underlying relationship pertaining to β_{L0} (short-term), γ_L (long-term), and β_{L2} (correlatedness) in equation (6). Since $\Delta Y_t := \frac{\theta_y}{\theta_x}\Delta X_t$ is the underlying linear relationship under the MIH (equation (10)) and $\beta_{L0} = \frac{\theta_y}{\theta_x}$ is the short-term effect, we have the following formal definition of a contagion.

Definition) *A contagion is declared if a long-term break (a break in* $\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$*) or a correlatedness break (a break in* $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$*) is mainly caused by a short-term break (a*

break in $\beta_{L0} = \frac{\theta_y}{\theta_x}$)[7].

This definition implies that a contagion is declared when the short-term effect of a shock is not controlled properly at the beginning of the crisis period and continues to cause breaks in long-term relationship or correlatedness between two markets.

First, we are concerned with finding whether a shock causes a break in the established linear relationship under the MIH. Therefore, we need to test whether a given period experiences breaks in the relationships pertaining to β_{M0} (short-term), γ_{M} (long-term), and β_{M2} (correlatedness) in equation (9). We develop a test adopting quantile regression and the *Z*-test for testing such breaks. Refer to Baur (2013) for more detailed discussions about the advantages of using quantile regression. Our test is based on the idea that a random fluctuation of the slope estimates around a constant value (with only the intercept parameters systematically increasing as a function of quantile ϑ) provides evidence for the iid error of the classical linear regression under the null MIH specified by equation (9). If some of the slope coefficients change as a function of quantile $0 \leq \vartheta \leq 1$, then we detect it via the *Z*-test, which is designed to assess the magnitude of the estimated change in slope for a given quantile. To test the null MIH, we establish the following hypotheses.

H_0 : $\beta_{\mathrm{L0}} = \beta_{\mathrm{M0}},\ \beta_{\mathrm{L1}} = \beta_{\mathrm{M1}},\ \beta_{\mathrm{L2}} = \beta_{\mathrm{M2}}$ ($\beta_{\mathrm{L0}} = \beta_{\mathrm{M0}},\ \gamma_L = \gamma_{\mathrm{M}},\ \beta_{\mathrm{L2}} = \beta_{\mathrm{M2}}$, equivalently)

H_a : At least one of the followings holds;

$\beta_{\mathrm{L0}} \neq \beta_{\mathrm{M0}},\ \beta_{\mathrm{L1}} \neq \beta_{\mathrm{M1}},\ \beta_{\mathrm{L2}} \neq \beta_{\mathrm{M2}}$ ($\beta_{\mathrm{L0}} \neq \beta_{\mathrm{M0}}, \gamma_L \neq \gamma_{\mathrm{M}}, \beta_{\mathrm{L2}} \neq \beta_{\mathrm{M2}}$, equivalently)

The above null hypothesis H_0 illustrates that β_{M0}, γ_M, and β_{M2} are fixed with iid error $\varepsilon_{\mathrm{Mt}}$ as noted in equation (9). In other words, markets X and Y maintain $\beta_{L0} = \beta_{\mathrm{M0}}$, $\gamma_L = \gamma_{\mathrm{M}}$, and $\beta_{L2} = \beta_{\mathrm{M2}}$ under the MIH. If we reject H_0, then the MIH fails to hold. In this

[7] Note that a long-term break or a correlatedness break can be caused by changes in (ρ_x, ρ_y) rather than by changes in $\frac{\theta_y}{\theta_x}$. This case is not considered as a contagion here because a break in the long-term relationship is not caused by a break in the underlying basic linear relationship (or a short-term break).

case, the markets fail to maintain their underlying relationship and experience breaks in the short- or long-term relationship or correlatedness.

To implement the test, we estimate the quantile regression parameters across the entire range of conditional quantiles of ΔY_t given ΔX_t. We estimate $(\beta_{\mathrm{L0}}, \gamma_{\mathrm{L}}, \beta_{\mathrm{L2}})$ using the quantile regression across fixed N quantiles. Let $\left(\hat{\beta}_{\mathrm{L0}j}, \hat{\gamma}_{\mathrm{L}j}, \hat{\beta}_{\mathrm{L2}j}\right)$ be the slope estimates of $\left(\beta_{\mathrm{L0}j}, \gamma_{\mathrm{L}j}, \beta_{\mathrm{L2}j}\right)$ from the $\frac{j}{N+1}th$ quantile regression for $j = 1, \cdots, N$. Using the Proposition B in the Appendix B, one may derive the Z-test as follows. Let P_i be $(\hat{\varphi}_{i1}, \cdots, \hat{\varphi}_{iN})$, where $\hat{\varphi}_{ij} (i = 1,2,3,\ j = 1, .., N)$ is the ith slope estimate for one of the three slope estimates ($\hat{\beta}_{\mathrm{L0j}}$, $\hat{\gamma}_{\mathrm{L}j}$, $\hat{\beta}_{\mathrm{L2}j}$) from the $\frac{j}{N+1}th$ quantile regression. Let $P_{i,-k}$ represent a vector constructed by excluding the kth element from $P_i = (\hat{\varphi}_{i1}, \cdots, \hat{\varphi}_{iN})$. For example, $P_{i,-1} = \left(\hat{\varphi}_{i2}, \cdots, \hat{\varphi}_{i,N}\right)$ and $P_{i,-N} = \left(\hat{\varphi}_{i1}, \cdots, \hat{\varphi}_{i,N-1}\right)$. Then,

$$Z_{i,k} = \frac{\hat{\varphi}_{ik} - m_k}{s_k} \text{ is N(0,1) asymptotically}^{8} \tag{12}$$

with mean m_k and standard deviation $\mathrm{s_k}$ from $P_{i,-k}$. This Z-test can be employed to test the following hypotheses.

H_0^{SB}: $\hat{\beta}_{\mathrm{L0}k}$ originates from the same normal distribution as the others in P_1 under H_0 (or there is no short-term break between X and Y in the $\frac{k}{N+1}th$ quantile).

H_1^{SB}: $\hat{\beta}_{\mathrm{L0}k}$ does not originate from the same normal distribution as the others in P_1 under H_{a} (or there is a short-term break between X and Y in the $\frac{k}{N+1}th$ quantile).

H_0^{LB}: $\hat{\gamma}_{\mathrm{L}k}$ originates from the same normal distribution as the others in P_2 under H_0 (or there is no long-term break between X and Y in the $\frac{k}{N+1}th$ quantile).

[8] This holds based on Proposition B in Appendix B and the weak law of large numbers under H_0.

H_1^{LB}: $\hat{\gamma}_{\mathrm{L}k}$ does not originate from the same normal distribution as the others in P_2 under H_{a} (or there is a long-term break between X and Y in the $\frac{k}{N+1}th$ quantile).

H_0^{CRB}: $\hat{\beta}_{\mathrm{L}2k}$ originates from the same normal distribution as the others in P_3 under H_0 (or there is no correlatedness break between X and Y in the $\frac{k}{N+1}th$ quantile).

H_1^{CRB}: $\hat{\beta}_{\mathrm{L}2k}$ does not originate from the same normal distribution as the others in P_3 under H_{a} (or there is a correlatedness break between X and Y in the $\frac{k}{N+1}th$ quantile).

As a break in the established relationship is expected to occur at low quantiles during crisis periods, a contagion test can use the slope estimates at low quantiles, namely $\hat{\beta}_{\mathrm{L}01}$, $\hat{\gamma}_{\mathrm{L}1}$, or $\hat{\beta}_{\mathrm{L}21}$. Selection of the low quantile might be justified by the fact that during the crisis period when ΔX_t decreases, ΔY_t tends to decrease more significantly than usual. Hence, a significant change in $\frac{\theta_{\mathrm{y}}}{\theta_{\mathrm{x}}}$ at low quantile is expected (recall $\Delta Y_t := \frac{\theta_{\mathrm{y}}}{\theta_{\mathrm{x}}}\Delta X_t$ under MIH by equation (10)). In summary, quantile regression is employed here to handle the contagion as a low tail event. By rejecting H_0^{SB}, H_0^{LB}, or H_0^{CRB} using $\hat{\beta}_{\mathrm{L}01}$, $\hat{\gamma}_{\mathrm{L}1}$, or $\hat{\beta}_{\mathrm{L}21}$, respectively, we can conclude breaks in the short-term relationship, long-term relationship, or correlatedness during crisis periods, respectively.

B. Contagion dynamics against the MIH

From the null MIH and hence equation (9), various kinds of situations can occur when a shock hits the market X. Changes in $(\frac{\theta_y}{\theta_x}, \rho_x, \rho_y)$ (and hence volatility spillover by equation (11)) caused by a shock bring about structure breaks in short- or long-term effects or correlatedness (β_{L0},γ_{L}, β_{L2}). There are eight possible cases in terms of breaks in short- or long-term effects or correlatedness (β_{L0}, γ_{L}, β_{L2}), resulting from the changes in $(\frac{\theta_y}{\theta_x}, \rho_x, \rho_y)$. Table 1 summarizes the eight possible situations (S1-S8) with their causality relations between $(\frac{\theta_y}{\theta_x}, \rho_x, \rho_y)$ and (β_{L0}, γ_{L}, β_{L2}) when a shock hits the market X. In the Appendix C,

we verify the causality relations of eight situations in Table 1. In the last column of Table 1, contagion check is addressed. "No" and "Contained"[9] indicate escape of contagion. "Additional check required" indicates that a further check is needed to conclude a contagion.

Table 1 Contagion dynamics table

This table shows eight possible situations that can occur when a shock hits the market X under equation (9). It also shows the causality relations between volatility change and structure break due to a shock to market X. The letters "C" and "NC" in the Causal volatility columns denote "change" and "no change," respectively. The "UND" denotes that "change" or "no change" is possible, but it is undecided. The letters "SB," "LB," "CRB," "NSB," "NLB," and "NCRB" in the Structure break columns denote short-term break, long-term break, correlatedness break, no short-term break, no long-term break, and no correlatedness break, respectively. The last column shows a contagion check where "No" and "Contained" indicate escape of contagion while "Additional check required" indicates that a further check is needed to conclude a contagion.

Classifi-cation	Causal volatility			Structure break			Contagion check
	$\frac{\theta_y}{\theta_x}$	ρ_x	ρ_y	Short-Term Effect (β_{L0})	Long-Term Effect (γ_L)	Correlated-ness (β_{L2})	
S-1	NC	NC	NC	NSB	NLB	NCRB	No
S-2	NC	C	C	NSB	NLB	CRB	No
S-3	NC	NC	C	NSB	LB	NCRB	Contained
S-4	C	C	NC	SB	NLB	NCRB	Contained
S-5	NC	C	UND	NSB	LB	CRB	Contained
S-6	C	UND	UND	SB	NLB	CRB	Additional check required
S-7	C	C	UND	SB	LB	NCRB	Additional check required
S-8	C	UND	UND	SB	LB	CRB	Additional check required

[9] Difference between "NO" and "Contained" is whether volatility spillover given by equation (11) occurs. Refer to Verifications of Table 1 in Appendix C.

Remark 5 (Contagion test bias). As noted in Table 1, a one-to-one relation between causal volatility vs structure break exists for S-1 to S-4 (i.e., it contains no UND), while no such one-to-one relation exists for S-5 to S-8 (i.e., it contains UNDs)[10]. The case of S-1 covers the case of normal MIH. The case of S-2 handles the contagion bias from correlatedness break (CRB) about which Forbes and Rigobon (2002) are most concerned. The case of S-3 handles the contagion bias from long-term break (LB). The case of S-4 handles the bias from short-term break (SB). This finding is related to the study of Favero and Giavazzi (2002) where they employ a nonlinearity technique to detect a short-term effect in its opposite direction to observe 'flight to quality'. The case of S-5 contains contagion via exclusively keeping $\frac{\theta_y}{\theta_x}$; hence, it might be considered as handling contagion bias from long-term break (LB) and correlatedness break (CRB) simultaneously. Note that S-5 is unlikely to occur in reality. The cases of S-6, S-7 and S-8 are basically concerned about the possible bias from short-term break (SB) as noted in S-4. In these cases, further checks are necessary because SB and CRB or SB and LB occur simultaneously.

C. Identifying contagion using a Bayesian approach

The change in underlying linear relationship or short-term effect $\beta_{L0} = \frac{\theta_y}{\theta_x}$ is further checked to determine whether it is the main force behind a break in long-term relationship (LB or CRB). Actually, the *Bayesian approach* is invoked to calculate the related posterior probabilities. Following the definition of the contagion, the occurrence of a contagion can be investigated by calculating the posterior probabilities of $P(SB|LB)$ *or* $P(SB|CRB)$. The posterior probability of $P(SB|LB)$ (P$(SB|CRB)$) is interpreted as a probability that LB (CRB) is caused by SB when LB (CRB) is given or as the ratio that SB explains LB (CRB). For S-1, S-2, S-3 and S-5, $P(SB|LB) = P(SB|CRB) = 0$ because they belong to NSB. For S-4, $P(SB|LB) = P(SB|CRB) = 0$ again because it belongs to NLB and NCRB simultaneously.

[10] Although causal volatility parts for S-6 and S-8 are identical, they are technically different in the sense that the UND for S-6 requires additional causal volatility restrictions for ρ_x and ρ_y. See verification of Table 1 in Appendix C.

Thus, either no or contained contagion is reported for S-1 to S-5.

For S-6, one needs to calculate $\mathrm{P(SB|CRB)} > 0$ (i.e., it belongs to SB and CRB). Since correlatedness $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ consists of $\frac{\theta_y}{\theta_x}$ and $(1-\rho_x)$, an additional check is necessary to decide whether CRB is mainly due to the change in $\frac{\theta_y}{\theta_x}$ (SB) or the change in $(1-\rho_x)$. The posterior probability that a correlated break (CRB) is caused by the changes in $\frac{\theta_y}{\theta_x}$ (or SB) is calculated by $\mathrm{P}(SB|CRB) = \frac{P(SB)P(CRB|SB)}{P(CRB)}$. For S-7, one needs to calculate $\mathrm{P(SB|LB)} > 0$ (i.e., it belongs to SB and LB). Since the long-term effect $\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ consists of $\frac{\theta_y}{\theta_x}$ and $\frac{(1-\rho_x)}{(1-\rho_y)}$, an additional check is necessary to decide whether LB is mainly due to the change in $\frac{\theta_y}{\theta_x}$ (SB) or the change in $\frac{(1-\rho_x)}{(1-\rho_y)}$. The posterior probability that the long-term break (LB) is caused by the changes in $\frac{\theta_y}{\theta_x}$ (or SB) is calculated by $\mathrm{P}(SB|LB) = \frac{P(SB)P(LB|SB)}{P(LB)}$. For S-8, one needs to further check that LB or CRB are due to the change in $\frac{\theta_y}{\theta_x}$ by calculating the posterior probabilities of $P(SB|LB)$ and $P(SB|CRB)$.

Our contagion test procedure is summarized as follows:

(i) *Assume the model (6) for target market Y and source market X,*

$$\Delta Y_t = \frac{\theta_y}{\theta_x}\Delta X_t - (1-\rho_y)\left(Y_{t-1} - \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}X_{t-1}\right) + \varepsilon_{Lt}$$
$$= \beta_{L0}\Delta X_t + \beta_{L1}(Y_{t-1} - \gamma_L X_{t-1}) + \varepsilon_{1t} = \beta_{L0}\Delta X_t + \beta_{L1}Y_{t-1} + \beta_{L2}X_{t-1} + \varepsilon_{Lt}$$

(ii) *Establish the null hypotheses for coefficients in the* $\frac{k}{N+1}th$ *quantile that corresponds to the crisis period across fixed N quantiles:* H_0^{SB} *vs* H_1^{SB} *(short-term break),* H_0^{LB} *vs* H_1^{LB} *(long-term break), and* H_0^{CRB} *vs.* H_1^{CRB} *(correlatedness break).*

(iii) *Apply Z-test given by equation (12) with selected* k *for* $(\beta_{L0}, \gamma_L, \beta_{L2})$.

(iv) Using the test results from step (iii) and Table 1, complete the Bayesian test by calculating posterior probabilities of P(SB│LB) and P(SB│CRB). If the posterior probability is greater than a specified value θ_0*, a contagion is declared.*

For calculating posterior probabilities of $P(SB|LB)$ *and* $P(SB|CRB)$, the p-values from step *(iii)* of the contagion test process are used to estimate the related probabilities. We use the p-values from testing H_0^{SB}, H_0^{LB} and H_0^{CRB} as estimates of $P(SB), P(\mathrm{L}B)$ and $P(\mathrm{CR}B)$, respectively. Recalling that the p-value is the probability, under H_0, of obtaining a result equal to or more extreme than what was actually observed, $P(SB|LB)$ is interpreted as the posterior probability that LB is caused by SB when LB is given or as the ratio in which SB explains LB. Since $\gamma_L = \beta_{L0}\frac{1-\rho_x}{1-\rho_y}$ with independent multiplication components β_{L0} and $\frac{1-\rho_x}{1-\rho_y}$, it is logical to assume that the break in β_{L0} or $\frac{1-\rho_x}{1-\rho_y}$ causes the break in γ_L. Recalling that $\beta_{L2} = \beta_{L0}\,(1-\rho_x)$ with independent multiplication components β_{L0} and $(1-\rho_x)$, $P(SB|CRB)$ can be obtained in a similar fashion. For $\widehat{\mathrm{P}}(\mathrm{L}B|SB)$ and $\widehat{\mathrm{P}}(\mathrm{CR}B|SB)$, one may estimate them from given testing results, which test H_0^{SB}, H_0^{LB} and H_0^{CRB} across m different target markets Ys given the same source market X. Suppose that tests yielded the m_1 Ys having experienced SB, m_2 Ys having experienced LB, m_3 Ys having experienced CRB, m_{12} Ys having experienced SB and LB, and m_{13} Ys having experienced SB and CRB. Then, $\widehat{\mathrm{P}}(\mathrm{L}B|SB) = \frac{m_{12}}{\mathrm{m}_1}$ and $\widehat{\mathrm{P}}(\mathrm{CR}B|SB) = \frac{m_{13}}{\mathrm{m}_1}$. Using those estimates of conditional probabilities we obtain $\widehat{\mathrm{P}}_0(SB|LB) = \frac{\widehat{\mathrm{P}}(SB)\widehat{\mathrm{P}}(LB|SB)}{\widehat{\mathrm{P}}(LB)}$ and $\widehat{\mathrm{P}}_0(SB|CRB) = \frac{\widehat{\mathrm{P}}(SB)\widehat{\mathrm{P}}(CRB|SB)}{\widehat{\mathrm{P}}(CRB)}$. If $\widehat{\mathrm{P}}_0(SB|LB)$ or $\widehat{\mathrm{P}}_0(SB|CRB)$ exceeds one, the posterior probability is set equal to one. Thus, we have posterior probability estimators as follows:

$$\widehat{\mathrm{P}}(SB|LB) = min\left(\widehat{\mathrm{P}}_0(SB|LB), 1\right)$$

$$\widehat{\mathrm{P}}(SB|CRB) = min\left(\widehat{\mathrm{P}}_0(SB|CRB), 1\right)$$

IV. Conclusion

In the literature, contagion tests are known to suffer from various biases, which started with 'interdependence, not contagion issue' as noted by Forbes and Rigobon (2002). Considerable studies have been performed to remove confounding effects or contagion biases from the comovement or correlation in excess. Solutions to the two critical issues have been pursued by employing various models. This paper resolves these critical issues by linking LFM and SEECM and applying the Bayesian approach, which determines the cause of the breaks in long-term relationship or correlatedness. Our contagion test is based on a more concrete definition of contagion, which is given by a *long-term or correlatedness break mainly caused by a short-term break*, and easy to implement. From a statistical point of view, the correlation bias issue here is closely related to the well-known statistical issue of correlation versus causation. In this paper, we overcome the correlation bias issue by accurately determining the cause of excessive correlation using a Bayesian approach.

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Appendix A. Proof of Proposition 1.

The following may easily be derived from equation (1)

$$\Delta Y_t = \frac{\theta_y}{\theta_x}\Delta X_t - \left(Y_{t-1} - \frac{\theta_y}{\theta_x}X_{t-1}\right) - \frac{\theta_y}{\theta_x}\delta_x u_{x,t} + \delta_y u_{y,t}$$

Then, it is easy to verify that

$$\begin{aligned}
\Delta Y_t &= \frac{\theta_y}{\theta_x}\Delta X_t - (1-\rho_y)\left(Y_{t-1} - \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}X_{t-1}\right) - \frac{\theta_y}{\theta_x}\delta_x u_{x,t} + \delta_y u_{y,t} - \rho_y Y_{t-1} \\
&\quad + \rho_x \frac{\theta_y}{\theta_x}X_{t-1} \\
&= \frac{\theta_y}{\theta_x}\Delta X_t - (1-\rho_y)\left(Y_{t-1} - \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}X_{t-1}\right) - \frac{\theta_y}{\theta_x}\delta_x u_{x,t} + \delta_y u_{y,t} - \rho_y(\theta_y W_{t-1} \\
&\quad + \delta_y u_{y,t-1}) + \rho_x \frac{\theta_y}{\theta_x}(\theta_x W_{t-1} + \delta_x u_{x,t-1}) \\
&= \frac{\theta_y}{\theta_x}\Delta X_t - (1-\rho_y)\left(Y_{t-1} - \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}X_{t-1}\right) - \frac{\theta_y}{\theta_x}\delta_x a_{u,t} + \delta_y b_{u,t} - \rho_y\theta_y W_{t-1} \\
&\quad + \rho_x\theta_y W_{t-1}. \\
&= \frac{\theta_y}{\theta_x}\Delta X_t - (1-\rho_y)\left(Y_{t-1} - \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}X_{t-1}\right) + \delta_y b_{u,t} - \frac{\theta_y}{\theta_x}\delta_x a_{u,t} + (\rho_x - \rho_y)\theta_y W_{t-1}.
\end{aligned}$$

Thus $\epsilon_{Lt} = \delta_y b_{u,t} - \frac{\theta_y}{\theta_x}\delta_x a_{u,t} + (\rho_x - \rho_y)\theta_y W_{t-1}$. Now it is easy to see that $Var(\varepsilon_{Lt}) = \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2 + (\rho_x - \rho_y)^2{\theta_y}^2$ and $Cov(\varepsilon_{Lt}, \epsilon_{L(t+1)}) = 0$. The proof is complete. Q.E.D

Appendix B. Proposition B (Refer to (3.7) of Koenker (2005), p.73).

Under H_0, with some regular conditions, the quantile slope estimates $\widehat{\boldsymbol{\beta}}_{MN} = (\widehat{\beta}_{Mj})_{j=1}^{N}$ have an asymptotically multivariate normal distribution $N(\boldsymbol{\beta}_{MN}, n\mathbf{V})$ for $\boldsymbol{\beta}_{MN} = (\beta_j)_{j=1}^{N}$ and a given variance covariance matrix $\mathbf{V}_{3N\times 3N}$ as the number of observations n increases.

Appendix C. Verification of Table 1

Below, we assume that equation (6) always holds as general case (i.e., equation (9) from the null MIH is a special case of equation (6), with $\rho_x = \rho_y = \rho_0$). Based on equation (6), it is straightforward to observe that $\beta_{L0} = \frac{\theta_y}{\theta_x}$, $\gamma_{\mathrm{L}} = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$, and $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$. Acceptance of β_{L0}, γ_L, or β_{L2} in a selected quantile (or acceptance of H_0^{SB}, H_0^{LB}, or H_0^{CRB} equivalently) indicates no change in β_{L0}, γ_L, or β_{L2} (or NSB, NLB, or NCRB) in the quantile, respectively, when a shock hits market X. For each situation, we discuss possible volatility spillover.

(S-1) Acceptance of $\beta_{L0} = \frac{\theta_y}{\theta_x}$ (short-term), $\gamma_{\mathrm{L}} = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ (long-term) and $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ (correlatedness) together implies no change in $\frac{\theta_y}{\theta_x}$ and $\rho_x = \rho_y = \rho_0$. In this case, there is no volatility spillover at all by equation (11). Thus, S-1 follows. Clearly, S-1 escapes the contagion, and the contagion check is "NO".

(S-2) Acceptance of $\beta_{L0} = \frac{\theta_y}{\theta_x}$ (short-term) and rejection of $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ (correlatedness) implies no change in $\frac{\theta_y}{\theta_x}$ and change of ρ_x (or ρ_x'). Acceptance of $\gamma_{\mathrm{L}} = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ (long-term) necessarily implies a change of ρ_y(or ρ_y') such that $\rho_x' = \rho_y'$. Thus, S-2 follows. In this case, there is no volatility spillover at all by equation (11), that is,

$$Var(\varepsilon_{\mathrm{Lt}}) = \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2 + \left(\rho_x' - \rho_y'\right)^2\theta_y^2 = \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2 = \mathrm{Var}(\varepsilon_{Mt}).$$

Clearly, S-2 escapes contagion, and the contagion check is "NO".

(S-3) Acceptance of $\beta_{L0} = \frac{\theta_y}{\theta_x}$ (short-term) and $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ (correlatedness) implies no change in $\frac{\theta_y}{\theta_x}$ and ρ_x. The rejection of $\gamma_{\mathrm{L}} = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ (long-term) necessarily implies a change of ρ_y(say, ρ_y'). Thus, S-3 follows. In this case, there is volatility spillover

$$Var(\varepsilon_{\mathrm{Lt}}) = \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2 + \left(\rho_x - \rho_y'\right)^2\theta_y^2 > \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2 = \mathrm{Var}(\varepsilon_{Mt})$$

and

$$\Delta Y_t = \frac{\theta_y}{\theta_x}\Delta X_t - \left(1 - \rho_y'\right)\left(Y_{t-1} - \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y')}X_{t-1}\right) + \varepsilon_{Lt}.$$

Since a break in the basic linear relationship does not occur (or NSB), S-3 escapes the contagion even though there is volatility spillover and the contagion check is "Contained". In this situation, market Y maintains market stability via controlling ρ_y' (or its own market-specific volatility) properly.

(S-4) Rejection of $\beta_{L0} = \frac{\theta_y}{\theta_x}$ (short-term) and acceptance of $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ (correlatedness) necessarily implies changes in $\frac{\theta_y}{\theta_x}$ and ρ_x (or $\frac{\theta_y'}{\theta_x'}$ and ρ_x'). Acceptance of $\gamma_{\mathrm{L}} = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ (long-term) and $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ (correlatedness) implies no change in ρ_y. Thus, S-4 follows. In this case, there might be volatility change by equation (11), that is,

$$\mathrm{Var}(\varepsilon_{\mathrm{Lt}}) = \delta_y^2 + \left(\frac{\theta_y'}{\theta_x'}\delta_x\right)^2 + \left(\rho_x' - \rho_y\right)^2\theta_y'^2 \neq \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2 = \mathrm{Var}(\varepsilon_{Mt})$$

and

$$\Delta Y_t = \frac{\theta_y'}{\theta_x'}\Delta X_t - \left(1 - \rho_y\right)\left(Y_{t-1} - \frac{\theta_y'(1-\rho_x')}{\theta_x'(1-\rho_y)}X_{t-1}\right) + \varepsilon_{Lt}.,$$

Since a break in the long-term relationship does not occur (or NLB and NCRB), S-4 escapes the contagion even though there could be volatility spillover and the contagion check is "Contained". In this situation, market Y maintains market stability via keeping ρ_y unchanged.

(S-5) Acceptance of $\beta_{L0} = \frac{\theta_y}{\theta_x}$ (short-term) and rejection of $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$

(correlatedness) necessarily implies no change in $\frac{\theta_y}{\theta_x}$ and a change in ρ_x(or ρ'_x). In this case, rejection of $\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ (long-term) do not produce any required situation for ρ_y (i.e., UND) but $\rho'_y \neq \rho'_x$. Thus, S-5 follows. In this case, volatility spillover clearly occurs by equation (11) as follows:

$$Var(\varepsilon_{Lt}) = \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2 + \left(\rho'_x - \rho'_y\right)^2\theta_y^2 > \delta_y^2 + \left(\frac{\theta_y}{\theta_x}\delta_x\right)^2 = \mathrm{Var}(\varepsilon_{Mt})$$

and

$$\Delta Y_t = \frac{\theta_y}{\theta_x}\Delta X_t - \left(1-\rho'_y\right)\left(Y_{t-1} - \frac{\theta_y(1-\rho'_x)}{\theta_x(1-\rho'_y)}X_{t-1}\right) + \varepsilon_{Lt}.$$

Since a break in the basic linear relationship does not occur (or NSB), S-5 escapes the contagion even though there is volatility spillover and the contagion check is "Contained". In this situation, market Y maintains market stability by keeping $\frac{\theta_y}{\theta_x}$ unchanged.

(S-6) Rejection of $\beta_{L0} = \frac{\theta_y}{\theta_x}$ (short-term) implies a change in $\frac{\theta_y}{\theta_x}$. In this case, acceptance of $\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ (long-term) and rejection of $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ (correlatedness) produce some required condition for ρ_y and ρ_x (e.g., $\frac{\theta'_y(1-\rho'_x)}{\theta'_x(1-\rho'_y)} = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$) although they are allowed to vary. Thus, (S-6) follows. Since the statuses of ρ_x and ρ_y are unknown (i.e., UND) with breaks in short-term relationship and correlatedness (or SB and CRB), this case needs a further check to conclude contagion, and the contagion check is "Additional check required". Following the definition of the contagion, a contagion occurs if the correlatedness break (or break in $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$) is mainly caused by a change in $\frac{\theta_y}{\theta_x}$ not by the change in $(1-\rho_x)$.

(S-7) Rejection of $\beta_{L0} = \frac{\theta_y}{\theta_x}$ (short-term) and acceptance of $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ (correlatedness) necessarily implies a change in $\frac{\theta_y}{\theta_x}$ and ρ_x (or $\frac{\theta'_y}{\theta'_x}$ and ρ'_x). In this case,

rejection of $\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ (long-term) does not produce any required situation for ρ_y. Thus, S-7 follows. Since the status of market-specific volatility of Y (ρ_y) is unknown (i.e., UND) with breaks in short and long-term relationship (or SB and LB), this case needs a further check to conclude contagion, and the contagion check is "Additional check required". Following the definition of the contagion, a contagion occurs if the long-term break (or break in $\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$) is mainly caused by the change in $\frac{\theta_y}{\theta_x}$ not by the change either in $(1-\rho_x)$ or $(1-\rho_y)$.

(S-8) Rejection of $\beta_{L0} = \frac{\theta_y}{\theta_x}$ (short-term) implies a change in $\frac{\theta_y}{\theta_x}$ (or $\frac{\theta_y'}{\theta_x'}$). It is trivial to see that rejection of $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$ (correlatedness) and rejection of $\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ (long-term) do not produce any required situation for ρ_y and ρ_x. Thus, S-8 follows. Since the statuses of ρ_x and ρ_y are unknown (i.e., UND) with breaks in short- and long-term relationship and correlatedness (or SB, LB, and CRB), this case needs a further check to conclude a contagion, and the contagion check is "Additional check required". Following the definition of the contagion, a contagion occurs if the long-term break or correlatedness break (or break in $\gamma_L = \frac{\theta_y(1-\rho_x)}{\theta_x(1-\rho_y)}$ or break in $\beta_{L2} = \frac{\theta_y(1-\rho_x)}{\theta_x}$) is mainly caused by the change in $\frac{\theta_y}{\theta_x}$ not by the change either in $(1-\rho_x)$ or $(1-\rho_y)$.