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Steven Kou[†] and Seyoung Park[‡]

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Keywords: Income Shock, Incomplete Market, Consumption Smoothing, Precautionary Savings, General Equilibrium

JEL Codes: D15, D58, G11, G12

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*The authors are grateful for the helpful discussions with Darrell Duffie, Paul Glasserman, Jussi Keppo, Karl Schmedders, Alan Morrison, David Bell, Abhay Abhyankar, Alain Bensoussan, Chiaki Hara, Eckhard Platen, Paul Embrechts, Hyeng Keun Koo, Phillip Yam, Huainan Zhao, and Jiro Akahori. All errors are the authors' own responsibility.

1 Introduction

1.1 Background

We are experiencing the unstoppable technological revolution and emerging new forms of work driven by artificial intelligence and big data analysis. Frey and Osborne (2017) demonstrate that 47% of jobs in the U.S. runs the risk of being automated, thereby disrupting labor markets and affecting workers negatively in the long run. Potentially catastrophic loss of income is an omnipresent risk. It increases an individual's concern about future income uncertainty, so many people face the challenge of determining how to continue being able to afford what they can currently afford, i.e., how to attain a smooth profile of future consumption.[1]

Our focus is to generalize Friedman's (1957) permanent income hypothesis (PIH) with a large, negative income shock (LNIS) and then to evaluate whether the generalized framework can explain how people would respond to the LNIS to increase their resilience to it through the lens of their precautionary savings.[2] Having generalized the PIH with the

[1] In the US, 44% of households could not cover an emergency expense of only $400, so they will struggle if they experience an unexpected hardship (Federal Reserve report, 2017). For more details, refer to "Report on the Economic Well-Being of U.S. Households in 2016" published by Board of Governors of the Federal Reserve System on May 2017. In the European Union, approximately 218 million people are experiencing earnings insecurity and volatility and struggling to ensure that future consumption needs can be met (European Commission statistics, 2017). For more details, refer to EU Statistics on Income and Living Conditions (EU-SILC 2017).

[2] As to individual savings behavior, the substantial portion of US and China households savings is driven by precautionary reasons (Choi *et al.*, 2017). In our equilibrium model, the precautionary savings could

LNIS, we then examine its effect on interest rates by using a general equilibrium analysis. For analytically tractable general equilibrium analysis, we consider a pure exchange economy in the type of Lucas (1978). The generalization is that a representative agent's aggregate output is exposed to the LNIS. To investigate the effects of the LNIS in our equilibrium setting, we have derived Euler equation and general equilibrium quantities in closed-form.

1.2 Contribution

We make the following two contributions in this paper.

- The precautionary savings could increase as wealth increases (Figure 2 in Section 5).
- The LNIS could influence a decrease in interest rate (Theorem 4.2 in Section 4; Figure 5 in Section 5).
- The LNIS significantly improves our equilibrium model's ability to match the equity premium and risk-free rate of the century-long sample (1891-1994) (Figure 7 in Section 5).

Firstly, we generalize the PIH to include the LNIS and quantify the required amount of savings that should be amassed for precautionary reasons (or precautionary savings). Contrary to the predictions of Bewley (1977) and Campbell (1987), an income shock would have significant effects on the precautionary savings of rich people. We find that the

correspond to a decrease in the subjective discount rate, implying a high savings rate and low interest rate.

possibility of the LNIS increases the precautionary savings as wealth increases, thereby explaining high savings rates of the wealthy rather than negative savings rates predicted by Bewley (1977) and Campbell (1987). Overall, relative to low-wealth people, high-wealth people can save more thanks to their affordability to amass the required precautionary savings for consumption smoothing.

Secondly, in the derived Euler equation, we show that the amount of present consumption the agent would be willing to give up now to receive one more unit of future consumption becomes larger than without the LNIS. Indeed, in the presence of the LNIS, the derived Arrow-Debreu price (or shadow price) increases and its expected return decreases by the amount of risk premium that is associated with the LNIS. Accordingly, the agent's demand for precautionary savings is sufficiently strong making her save at a high rate and thus lowering the equilibrium interest rate significantly, which is particularly relevant to today's low-interest-rate environment.

Finally, the equity premium and risk-free rate are determined in equilibrium and matched up with the century-long sample (1891-1994). Reasonable parameters of risk aversion generate a 3% to 6% equity premium and a 1% to 2.5% risk-free rate. Compared to the standard asset pricing model without the LNIS, ours with the LNIS generates a significantly low risk-free rate even with very small possibilities of the LNIS. Further, contrast to the standard model, in our model high values of risk aversion no longer counterfactually generate a high risk-free interest rate, thereby avoiding the risk-free rate puzzle (Weil, 1990). Rather, an increase in risk aversion can result in a decrease in risk-free rate with the LNIS. Intuitively, the LNIS faced by individuals cause them likely to reduce consumption

to secure extra reserves as a precaution or to finance consumption needs by using their savings. Such a conservative decision discourages equity investment, so the equity premium increases and the risk-free rate decreases.

1.3 Literature Review and Outline

Permanent Income Hypothesis. Existing theories on household consumption cannot explain recent empirical findings on individual savings behavior. The permanent income hypothesis (PIH) of Friedman (1957) suggests that people should save now to prepare for the aftermath of a permanent decline in income (income shock), which has been formulated rigorously by Bewley (1977) and Campbell (1987). The PIH demonstrates that consumption is proportional to the sum of financial wealth and human capital, which is the discounted expected value of future income at the risk-free interest rate. So, the changes in marginal consumption with financial wealth are constant, thereby leading to consumption smoothing regardless of wealth levels. However, many people are not ready to meet their future consumption needs if an income shock occurs. Thus, when future disastrous income shocks are possible, Friedman's model may not be applicable.

Individual Savings. According to the PIH predictions of Bewley (1977) and Campbell (1987), an income shock is less likely to affect the precautionary savings of people who are at the higher end of wealth. This is because consumption of the wealthy can be financed mainly by wealth, without resorting to income. Intuitively, the ability to self-insure against

the income shock improves when wealth is large, so the precautionary savings decrease as wealth increases and then turns negative if wealth is large relative to income. However, empirical and anecdotal evidence shows that positive and even high savings rates are very common amongst wealthy people, and are they well known amongst financial advisors. Indeed, wealthy people are inclined to exhibit high savings rates as follows: a positive relation between savings rates and income (Dynan *et al.*, 2004), entrepreneurship purposes for entering and expanding business (Buera, 2009), out-of-pocket medical expenses patterns (De Nardi *et al.*, 2010), the mix of bequests and human capital, entrepreneurship, and medical-expense risk (De Nardi and Fella, 2017).

Optimal Consumption and Investment. Our work sits squarely within the optimal consumption and investment framework. Starting from the seminal work of Merton (1969, 1971), many studies have incorporated nontradable income in the framework.[3] Importantly, the (undiversifiable) labor income risk has become a standard element in studies of optimal strategies. When modeling and interpreting labor income shocks, standard literature has assumed log-normality with a Brownian motion.[4] However, large and negative income shocks affect individual life-cycle strategies significantly, and cause substantial deviations from log-normality (Guvenen *et al.*, 2015). Furthermore, models that use the Brownian

[3]For instance, refer to Farhi and Panageas (2007), Choi *et al.* (2008), and Jang *et al.* (2013).

[4]For example, Merton (1971), Bodie *et al.* (1992), Duffie *et al.* (1997), Koo (1998), Cocco *et al.* (2005), Gomes and Michaelides (2005), Polkovnichenko (2007), Benzoni *et al.* (2007), Wachter and Yogo (2010), Dybvig and Liu (2010), Munk and Sørensen (2010), Lynch and Tan (2011a, 2011b), Calvet and Sodini (2014), Ahn *et al.* (2019), and Jang *et al.* (2019).

motion cannot appropriately account for the effects of low-probability, high-impact events such as forced unemployment and job displacement.[5] To increase the realism of the model, we consider the LNIS in the optimal consumption and investment framework.

Incomplete Market. It has long been known that market completeness under no arbitrage implies the existence of a unique state price density and the resulting unique risk-neutral measure, under which the expected return on any asset becomes the risk-free rate (Ross, 1978). However, when markets are incomplete, i.e., when risk may be undiversifiable, the assumption of risk-neutrality with the unique state price density cannot be justified. Rather, the number of state price densities is infinite, so the set of equivalent martingale measures is also infinite. To price the expected return on an asset under the LNIS, this multitude of state price densities must be pruned to one.

Although a theoretical framework for the martingale pricing in incomplete markets exists (Karatzas *et al.*, 1991), it cannot easily be used to characterize the set of state price densities explicitly.[6] The state price density has been derived explicitly by Kou (2002) and

[5] Low et al. (2010) show that large earnings losses are observed at job displacement.

[6] In order to address the challenges of market incompleteness, instead of the martingale pricing approach, alternative dynamic programming approach can be used for the pricing in incomplete markets (Duffie *et al.*, 1997; Liu et al., 2005). However, in this case, it involves highly non-linear Hamilton-Jacobi-Bellman (HJB) equations, which are almost impossible to solve analytically. Therefore, use of dynamic programming approach requires use of complex numerical schemes to solve incomplete market problems. With no consideration of labor income and its risk, one can adopt the approaches of Garlappi and Skoulakis (2010), Jin and Zhang (2012), and Jin et al. (2017) for such a numerical approach to solving the consumption

Liu et al. (2003). However, these models have overlooked the labor income risk, which is a major dimension of market incompleteness. Based on the idea of market completion by Karatzas *et al.* (1991), Liu *et al.* (2003) have established a dynamically completed market with derivatives. Similar to Liu *et al.* (2003), Branger et al. (2017) add derivatives to complete the market in which variances and covariances of return are stochastic. It would be tricky to apply their models to the incomplete market setting, without resorting to the idealistically completed market.

In our work, the LNIS is driven by an exogenous shock that is assumed to occur with a Poisson probability distribution, and hence may not be fully diversified away. In such a setting with the random arrival of the income shock, the classical martingale pricing approach (Cox and Huang, 1989) that uses the risk-neutral measure is no longer available. We believe the main difficulty lies in that allowing for market incompleteness usually gives rise to considerable challenges in deriving the closed-form (or analytically tractable) unique state price density for pricing purposes. Ours is a first attempt to develop an analytically tractable martingale pricing approach in an incomplete market with the LNIS. We start from the idea of fictitious completion (Karatzas *et al.*, 1991) to establish a risk-neutral intensity of the LNIS; the uniquely determined risk-neutral intensity should be used when managing an individual's dynamic budget constraint.[7]

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and investment problem in incomplete markets.

[7]More recently, Ahn et al. (2019) provide a numerical scheme to characterize the unique state price density.

General Equilibrium Analysis. There is an extensive literature on a general equilibrium analysis. Existing equilibrium approaches of Basak (1995), Heaton and Lucas (1996), Basak and Cuoco (1998), Basak and Shapiro (2001), Liu et al. (2003), Maenhout (2004), Gârleanu and Panageas (2015), Kimball *et al.* (2018), Gomes (2019) assume market completeness and therefore cannot investigate the consequence of the LNIS. Wang (2003), Gomes and Michaelides (2008), Guvenen (2009), Krueger and Lustig (2010), Christensen *et al.* (2012), Dumas and Lyasoff (2012) have incorporated unhedgeable income shocks in their equilibrium analysis, but do not consider disastrous income shocks such as the LNIS. These models consider only diffusive-type income shocks and cannot account for jump-type income risk. Ours is the first study to consider both diffusive income shocks and LNIS in the equilibrium analysis, and thereby to explain today's low-interest-rate environment through the precautionary savings channel.[8]

Related Literature. Three papers share similarities with problems of a permanent discrete-jump income shock.[9] One (Wang *et al.*, 2016) studies an incomplete-market consumption-savings model with recursive utility and stochastic income modeled by both a

[8]According to the Ramsey rule (Ramsey, 1928), interest rate equals subjective discount rate plus the product of consumption elasticity of marginal utility and consumption growth rate. We generalize the Ramsey rule in a rational expectations equilibrium framework.

[9]Gormley *et al.* (2010) have considered a large, negative wealth shock in the optimal consumption and investment framework. They have demonstrated that the absence of sufficient social security and insurance against the wealth shock can explain the limited stock market participation puzzle and the low consumption and high savings rate puzzle.

Brownian motion and large jump income shocks. However, this paper neglects optimal investment in risky assets and does not investigate the effects of stochastic income on general equilibrium quantities. Another (Bensoussan *et al.*, 2016) studies an optimal retirement model that considers jump-type forced unemployment risk. However, this work uses a piecewise connected utility function to solve the case in which unbounded downside utilities are caused by forced unemployment risk.[10] Also, the paper does not analyze general equilibrium implications.[11] The other (Schmidt, 2016) studies asset pricing implications of idiosyncratic tail risk with recursive preferences, heterogeneous agents, and incomplete markets. However, Schmidt's work with different techniques distracts from our main point which is precautionary savings implications with the LNIS on interest rates. Further, different from Schmidt (2016), we are able to provide analytically tractable solutions for the optimal strategies and general equilibrium quantities.

Outline. This paper is organized as follows. In Section 2, we establish the optimal consumption and investment problem with the LNIS. In Section 3, we generalize the PIH and obtain analytically tractable optimal consumption and investment strategies, and thereby quantify the precautionary savings that are required to smooth consumption and demand

[10] With such piecewise connected utility function, they assume that people is risk averse for positive wealth, but indifferent for negative wealth up to their borrowing against labor income, which is not always justified in general cases.

[11] Technically, Wang *et al.* (2016) and Bensoussan *et al.* (2016) use the dynamic programming approach rather than the martingale pricing approach with the state price density, so their analyses is not straightforward to be extended to a general equilibrium analysis.

for hedging against the LNIS. In Section 4, we provide a general equilibrium analysis with a focus on interest rate. In Section 5, we perform an in-depth quantitative analysis to discuss various properties of the optimal strategies and general equilibrium quantities. In Section 6, we conclude the paper.

2 The Model

Model Primitives. We consider an infinite-horizon economy with a single consumption good (the numeraire). Each representative agent is assumed to derive utility from intermediate consumption in the form of

$$\int_0^\infty e^{-\beta t} U\big(c(t)\big) dt,$$

where $\beta > 0$ is the subjective discount rate, $c(\cdot)$ denotes reduced-form consumption of goods and services, and $U(\cdot)$ measures the agent's utility and is twice continuously differentiable, strictly increasing, and strictly concave.

Throughout the paper, we consider the following constant relative risk aversion (CRRA) utility function:

$$U\big(c(t)\big) = \frac{c(t)^{1-\gamma}}{1-\gamma},$$

where $c(t)$ is per-period consumption at time t, and $\gamma > 1$ is the constant coefficient for an agent's relative risk aversion.[12]

Uncertainty is driven by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, in which a multi-dimensional Brownian motion and a Poisson process are defined. All stochastic processes

[12] We assume that $\gamma > 1$, which is consistent with the data.

are adapted to $\{\mathcal{F}_t\}$, which is the P-augmentation of the filtration generated by the Brownian motion and the Poisson process. All stated stochastic processes are assumed to be well defined, without explicitly stating the regularity conditions.

Model Setup. Financial assets in the market are summarized by one riskless bond and multiple risky stocks. The bond price B and the stock prices S are given by

$$dB(t) = rB(t)dt$$

and

$$dS(t) + D(t) = S(t)\{\mu dt + \sigma^\top dZ(t)\},$$

where r is the risk-free interest rate, $D(t) = (d_1, ..., d_N)$ are dividends for N risky stocks, μ is the constant mean vector, σ is the constant nonsingular standard deviation matrix, and $Z(t)$ is the standard Brownian motion process with dimensionality equal to the number of linearly independent returns on stocks. In the context of finance interpretation, $Z(t)$ represents variations in market/economic condition that stem from a source of market risk (or a market factor) in the economy.

We consider aggregate output process $I(t)$ modeled by a geometric Brownian motion as:

$$dI(t) = \mu^I I(t)dt + (\sigma^I)^\top I(t)dZ(t), \quad I(0) = I > 0,$$

where μ^I is the output mean vector and σ^I is the standard deviation vector, and $Z(t)$ is the market factor. The output is exposed to a large, negative income shock (LNIS), which is assumed to be distributed according to an exponential distribution (i.e., a Poisson

shock) with intensity δ. Specifically, the output is assumed to plummet immediately to $kI(t)$ ($k \in (0,1)$) from $I(t)$ in the aftermath of such a random and jump income shock. We assume that the fraction $\xi \in (0,1)$ of aggregate output constitutes aggregate earnings $\xi I(t)$. The remaining fraction $1-\xi$ of aggregate output is paid out as a dividend as:

$$D(t) = (1-\xi)I(t).$$

The risk in the model results from two sources: the market factor and the undiversifiable LNIS. The market factor is captured by fluctuations $dZ(t)$ in market/economic conditions. The undiversifiable income shock is captured by a Poisson shock with a small probability that comes as the LNIS.

The Optimal Consumption and Investment Problem. This problem with the LNIS is given by:

$$V(w,I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\tau e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}V^A(W(\tau),I(\tau))\Big],^{13} \tag{1}$$

which is subject to the dynamic wealth constraint:

$$dW(t) = \{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\}dt + \pi(t)^\top\sigma^\top dZ(t), \quad W(0) = w > -\xi I/\beta_1, \tag{2}$$

$$\beta_1 = r - \mu^I + (\sigma^I)^\top\theta,$$

$$W(t) > -\frac{\xi I(t)}{\beta_1}, \quad \text{for all } 0 \le t < \tau,$$

[13]Throughout the paper, we only consider the set of admissible policies of consumption $c(t)$ and investment $\pi(t)$.

where τ is the random arrival of the LNIS, $V^A(W(\tau), I(\tau))$ is the case of Merton (1969, 1971) with a stochastic income stream $k\xi I(t)$ $(t \geq \tau)$ without the LNIS, $\mathbf{1}$ is a vector of one's with dimensionality equal to the number of stocks, and $\pi(t)$ is the dollar amount vector invested in each risky stock.

3 Generalized Permanent Income Hypothesis

$V^A(W(\tau), I(\tau))$ in (1) is known to have the following closed-form solution:

$$\begin{aligned} V^A(W(\tau), I(\tau)) &\equiv \sup_{(c,\pi)} E\Big[\int_\tau^\infty e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt\Big] \\ &= K\frac{\{W(\tau) + k\xi I(\tau)/\beta_1\}^{1-\gamma}}{1-\gamma}, \end{aligned}$$

where

$$K = A^{-\gamma},\ A = \frac{\gamma-1}{\gamma}\Big(r + \frac{||\theta||^2}{2\gamma}\Big) + \frac{\beta}{\gamma},\ \theta = (\sigma^\top)^{-1}(\mu - r\mathbf{1}).$$

Here, θ is the Sharpe ratio vector.

The optimal consumption and investment problem of (1) can be then rewritten as the following: after integrating out the Poisson intensity δ,[14]

$$V(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K\frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \tag{3}$$

subject to (2).

Given the restated optimal consumption and investment problem of (3), we can generalize Friedman's permanent income hypothesis (PIH) with the LNIS. We find that the

[14]The technical details of the problem derivation are available in an online appendix.

permanent discrete jump income shock makes the maximized expected discounted utility $-\infty$ (or the maximized expected utility $+\infty$). When people would choose to finance their future consumption needs by borrowing against their human capital, as suggested by the conventional permanent income hypothesis, the new intertemporal wealth maximization term caused by the income shock with δ on the right hand side of (3) could become $-\infty$, as wealth approaches the borrowing limit of $-\xi I(t)/\beta_1$. In this case, given $-\xi I(t)/\beta_1 < -k\xi I(t)/\beta_1$, the value function $V(w, I)$ also becomes $-\infty$. This technical challenge is acknowledged in Bensoussan *et al.* (2016) when a jump-type income shock occurs. We will address this challenge by newly imposing a lower bound of wealth.

The Optimal Consumption and Investment Strategies. The generalized PIH with the optimal consumption and investment framework should address the challenge of $-\infty$ driven by individual earnings catastrophe risk. To meet this requirement, we impose a catastrophically low time-varying value of wealth, reminiscent of a starvation level below which people cannot sustain themselves financially and consequently, do not invest in the stock market. The new lower bound of wealth can be given by:

$$W(t) > -L(t) > -\frac{k\xi I(t)}{\beta_1}, \quad \text{for all } t \geq 0, \tag{4}$$

where $L(t)$ is a given nonnegative time-varying function that makes endogenous investment in the stock market equal to zero.

Equation (4) enables derivation (in an online appendix) of analytically tractable optimal consumption and investment strategies with the LNIS.

Theorem 3.1 *The optimal strategies for consumption $c(t)$ and risky stock investment $\pi(t)$ are obtained in closed-form:*

$$c(t) = (\hat{A} + \delta)\Big[w + \frac{\xi I}{\beta_1} - \xi I B^*(\overline{z};\delta) z^{-\alpha_\delta^*} - PS\Big], \tag{5}$$

$$\begin{aligned}\pi(t) = {} & \frac{1}{\gamma}\sigma^{-1}\theta w \\ & + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\Big[\frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi I B^*(\overline{z};\delta) z^{-\alpha_\delta^*} \\ & - \frac{2\gamma}{||\beta_3||^2}\delta K \frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/ c(t)^{-\gamma} + (\gamma\alpha_\delta - 1)\times PS1 + (\gamma\alpha_\delta^* - 1)\times PS2\Big],\end{aligned} \tag{6}$$

where

$$\hat{A} = \frac{\gamma - 1}{\gamma}\Big(\beta_1 + \frac{||\beta_3||^2}{2\gamma}\Big) + \frac{\beta_2}{\gamma},$$

$B^*(\overline{z};\delta)$ and $\overline{z}$ are the two constants to be determined by the boundary conditions

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \quad \textit{and} \quad G'(\overline{z}) = 0,$$

$\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots to the following characteristic equation:

$$F(\alpha;\delta) \equiv -\frac{1}{2}||\beta_3||^2\alpha(\alpha - 1) + \alpha(\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0,$$

$G(z)$ satisfies the following non-linear differential equation:

$$\begin{aligned}-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) \\ + \beta_1 G(z) + \delta K\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \quad 0 < z < \overline{z},\end{aligned}$$

with

$$\beta_2 = \beta - \mu^I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,$$

$$\beta_3 = \gamma(\sigma^I)^\top - \theta^\top,$$

and PS represents the precautionary savings driven by the LNIS and it is given by

$$PS = PS1 + PS2,$$

$$PS1 = \frac{2\delta K(\alpha_\delta - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu < 0,$$

$$PS2 = \frac{2\delta K(\alpha_\delta^* - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu > 0.$$

Theorem 3.1 helps people to achieve consumption smoothing when the LNIS is possible. The classic PIH can be revisited as an annuitized version of consumption profiles over the life cycle:

$$c(t) = A\Big(w + \frac{\xi I}{\beta_1}\Big), \tag{7}$$

where

$$A = \frac{\gamma - 1}{\gamma}\Big(r + \frac{||\theta||^2}{2\gamma}\Big) + \frac{\beta}{\gamma},$$

results from (5) when $\sigma_I = 0$ and $\delta = 0$, i.e., with no income risk. According to this PIH, the changes in marginal consumption with financial wealth are constant, thereby leading to consumption smoothing regardless of wealth levels. In the case of income risk ($\sigma_I = 0$), but without the LNIS ($\delta = 0$), the individual who is borrowing constrained by (4) can achieve consumption smoothing by adopting the following consumption strategy:

$$c(t) = A\Big(w + \frac{\xi I}{\beta_1} - \xi I B^*(\overline{z}; 0) z^{-\alpha_0^*}\Big). \tag{8}$$

As precaution against income risk, the individual has savings that are given by the difference between (7) and (8):[15]

$$A\xi I B^*(\overline{z}; 0) z^{-\alpha_0^*}. \tag{9}$$

[15]People who are faced with extra risk are likely to finance current/future consumption by relying on their

Alternatively, precautionary savings can be calculated from total income risk that results from output standard deviation $\sigma_I > 0$ and the LNIS $\delta > 0$ as the difference between (7) and (5). Further, comparison of precautionary savings with and without the LNIS yields PS in Theorem 3.1; this quantity represents the precautionary savings driven by the LNIS. Our quantitative analysis in the next section uses these precautionary savings to determine how much should individuals be prepared now for their consumption smoothing.

The classic Merton (1969, 1971) investment rule can be revisited:

$$\pi(t) = \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) - \sigma^{-1}\sigma^I\frac{\xi I}{\beta_1}, \tag{10}$$

comes from (6) when $\sigma_I > 0$ and $\theta = 0$, i.e., with output uncertainty but with the LNIS. The first term on the right hand side of (10) represents the mean-variance asset allocation and the second one represents the demand for hedging (or the intertemporal hedging component) against the output uncertainty. The individual whose borrowing is constrained by (4) derives the following optimal investment strategy:

$$\pi(t) = \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) - \sigma^{-1}\sigma^I\frac{\xi I}{\beta_1} + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha_0^* - 1)\xi I B^*(\overline{z}; 0)z^{-\alpha_0^*}. \tag{11}$$

The difference between (10) and (11) can then capture the effects of borrowing constraints in (4):

$$-\frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha_0^* - 1)\xi I B^*(\overline{z}; 0)z^{-\alpha_0^*}. \tag{12}$$

savings or to cut down on consumption amount to reserve enough financial resources due to precautionary reasons (Campbell, 1987).

This quantity provides new understanding of individual investment with benchmarking. Specifically, when the benchmark is taken to be aggregate output, $\sigma^I\theta^{-1}$ can quantify the sensitivity of the benchmark (or the aggregate output) to economic conditions, and $1/\gamma$ can quantify the sensitivity of the Merton policy (or the stock investment) to economic conditions.[16] If the aggregate output becomes less sensitive to economic conditions than the stock investment, i.e., if $\sigma^I\theta^{-1} < 1/\gamma$, then the optimal behavior of the borrowing-constrained individual is to reduce the investment amount, and to rely on earnings generated by the aggregate output that constitute the implicit cash holdings in readiness for binding borrowing constraints. In an economy in which the aggregate output reacts more to changes in economic conditions than the stock investment, the individual earnings no longer act as a substitute for the implicit cash holdings, rather these earnings behave like a stochastic stream. Hence, the optimal choice is then to increase the investment amount, taking advantage of the risk premium and consequently, possibly avoiding binding borrowing constraints.

We generalize the Merton-type investment strategy with the LNIS ($\delta > 0$), thereby deriving the generalized optimal investment strategy in (6). Also, we can measure the demand of the borrowing-constrained individual for hedging against the LNIS as the difference between (10) and (6). The pure effects of the LNIS on investment, abstracting from borrowing constraints, are derived from the comparison of this difference with (12).

[16] Basak *et al.* (2006) have adopted these quantities in their analysis for the risk management with benchmarking.

Thus, the additional term in (6) given by

$$-\frac{2\gamma}{||\beta_3||^2}\delta K\frac{\left(w+\frac{k\xi I}{\beta_1}\right)^{1-\gamma}}{1-\gamma}\Big/c(t)^{-\gamma}$$

captures the demand for risk diversification by saving in risky assets. Specifically, it is the ratio of intertemporal wealth maximization term in (3), $\delta K(w+k\xi I/\beta_1)^{1-\gamma}/(1-\gamma)$, to the marginal utility of one extra unit of consumption, $c^{-\gamma}=(c^{1-\gamma}/(1-\gamma))'$. In the face of unexpected, large, and negative reductions in their earnings, people can sell their investment assets to provide funds for their consumption needs. The precautionary savings (PS) driven by the LNIS are also important in the investment. To attain consumption smoothing when the LNIS can occur, the optimal choice is to reduce the investment by some proportions $(\gamma\alpha_\delta-1)$ of $PS1$ and $(\gamma\alpha_\delta^*-1)$ of $PS2$. Overall, we find two opposing effects: first the LNIS encourages a risk-diversification motive that increases increases risky investment (Benzoni *et al.*, 2007; Ahn *et al.*, 2019); in contrast, the LNIS increases background risk and consequently increases the precautionary savings motive that reduces risky investment.

4 General Equilibrium Analysis

Equilibrium Building Blocks. We consider a simple pure exchange economy in the type of Lucas (1978). The economy is populated by a representative agent who encounters the LNIS. The agent is entitled to an endowment to be consumed in equilibrium and she is assumed to trade a riskless bond and multiple risky stocks distributing the dividend. The

returns to these assets adjust to represent a no-trade equilibrium.

Definition 4.1 *An equilibrium can be characterized as a collection of* (r, μ, σ) *and optimal strategies* $(c(t), \pi(t))$ *such that the consumption good, stock, and bond markets clear as*

$$c(t) = D(t),$$

$$\pi^j(t) = S^j(t)W(t), \quad j = 1, ..., N,$$

$$W(t) = \sum_{j=1}^{N} S^j(t),$$

where $D(t) = \sum_{j=1}^{N} d_j$ *is the aggregate dividend process and* N *is the number of risky stocks.*

State Price Density. The following theorem characterizes the unique state price density in the presence of income risk including the LNIS.

Theorem 4.1 *The unique state price density is derived in closed-form:*

$$\xi^{\hat{\delta}}(t) = \exp\Big\{ \ln\Big(\frac{\hat{\delta}}{\delta}\Big) \boldsymbol{1}_{\{\tau \le t\}} - (\hat{\delta} - \delta)t \Big\} H(t), \tag{13}$$

where

$$\hat{\delta} = \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{z},$$

τ *is the arrival time of a Poisson shock,* $\boldsymbol{1}$ *is an indicator function that gives* 1 *if the Poisson shock occurs at time* t *and* 0 *otherwise,* $G(z)$ *satisfies the following non-linear differential equation:*

$$-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z)$$
$$+ \beta_1 G(z) + \delta K \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \quad 0 < z < \overline{z},$$

with

$$\beta_2 = \beta - \mu^I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,$$

$$\beta_3 = \gamma(\sigma^I)^\top - \theta^\top,$$

the boundary conditions

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \quad \textit{and} \quad G'(\overline{z}) = 0,$$

and the dynamics of $H(t)$ *are given by*

$$dH(t) = -H(t)\{rdt + \theta^\top dZ(t)\}, \quad H(0) = 1.$$

Proof. See an online appendix. **Q.E.D.**

The unique state price density given in (13) is a generalized version of the well-known Arrow-Debreu price. The identified quantity $\xi^{\hat{\delta}}(t,\zeta)$ can be regarded as the Arrow-Debreu price per unit probability P of one unit consumption good in state $\zeta \in \Omega$ at time t.[17]

Theorem 4.1 allows a convenient multiplicative separation of the traditional Arrow-Debreu price and the LNIS adjustments. In the absence of the LNIS ($\delta = \hat{\delta}$), the state price density (13) reduces to the conventional Arrow-Debrew price, and presents only the output uncertainty adjustments. In the presence of the LNIS ($\delta \neq \hat{\delta}$), the extra income shock adjustments affect the generalized Arrow-Debreu price $\xi^{\hat{\delta}}(t)$ in (13).

[17]Basically, the Arrow-Debrew price is the future price of one unit consumption good. It can serve as a shadow price for discounting future costs and benefits in financial analysis. The derived unique state price density determines the risk-neutrality with respect to the LNIS. This price can be used as the Randon-Nikodym derivative for measure change purposes.

Equilibrium Consumption Price. The rational expectations equilibrium consumption price (equilibrium state price density) must satisfy the Euler equation (Stokey and Lucas, 1989).[18] When solving the optimal consumption and investment problem of (3) with dynamic wealth constraint (2) and lower bound of wealth (4), we can derive the following Euler equation:

$$U'(c(t)) = \lambda e^{(\beta-(\hat{\delta}-\delta))t} H(t).$$

Using Itô's formula, we explicitly derive the following equilibrium consumption dynamics:[19] for $t < \tau$,

$$\frac{dc(t)}{c(t)} = -\frac{U'(c(t))}{U''(c(t))c(t)}\Big[\Big\{\Big(r + (\hat{\delta} - \delta)\Big) - \beta\Big\}dt - \theta^\top dZ(t)\Big].$$

Therefore, the Euler equation shows that equilibrium consumption growth is measured as a gap between income-risk-adjusted interest rate of $(\hat{\delta} - \delta)$ and subjective discount rate of β, weighted by the elasticity of intertemporal substitution. The risk-neutral Poisson intensity $\hat{\delta}$ is included in addition to the original Poisson intensity δ. $\hat{\delta}$ either increases or decreases the equilibrium consumption growth relative to risk-free interest rate r by the amount of $\hat{\delta} - \delta$. When $\hat{\delta} = \delta$, the LNIS can be fully diversified; as a result, it is rewarded with the zero risk premium. However, when $\hat{\delta} \neq \delta$, the LNIS cannot be diversified, so it should be compensated for by the nonzero risk premium $\hat{\delta} - \delta$.[20] In the context of the

[18] An Euler equation is a differential equation that represents an intertemporal first-order condition for optimal consumption (Durlauf and Blume (2008), pp. 1854-1855).

[19] This dynamics are equivalent to the dynamics of equilibrium consumption price $e^{-\beta t}\xi^{\hat{\delta}(t)}$ for $t < \tau$.

[20] The canonical CAPM does not generate the nonzero income risk premium as we obtained, whereas in this paper, the LNIS can be thought of as extra undiversifiable risk source, accordingly, it should be

traditional risk-return trade-off (i.e., high risk and high return), $\hat{\delta}$ should be larger than δ,[21] as a result, the income risk premium should be positive. Thus, the effective risk-adjusted interest rate represents such a risk compensation for additional exposure to the LNIS, and thereby increases both equilibrium consumption growth and equilibrium consumption price compared to the case without the LNIS.[22]

Equilibrium Risk-Free Interest Rate. The following theorem solves the equilibrium risk-free interest rate.

Theorem 4.2 *The equilibrium risk-free interest rate is derived in closed-form:*

$$r = \beta + \gamma\mu^I - \frac{1}{2}\gamma(1+\gamma)(\sigma^I)^2 - (\hat{\delta}(r) - \delta), \tag{14}$$

where μ^I and σ^I represent the expected consumption growth rate and volatility of consumption growth rate, and the constant $\hat{\delta}(r)$ is determined by solving the following non-linear algebraic equation:

$$\hat{\delta}(r) = \Big\{\Big(\frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))}\Big)\Big/\Big(\frac{w}{\xi I} + \frac{k}{\beta_1(\hat{\delta}(r))}\Big)\Big\}^{\gamma}\{\beta_1(\hat{\delta}(r))\}^{\gamma}\delta K(r)$$

with

$$\beta_1(\hat{\delta}(r)) = \beta + (\gamma-1)\mu^I - \frac{1}{2}\gamma(\gamma-1)(\sigma^I)^2 - (\hat{\delta}(r) - \delta),$$

priced.

[21] This result can be also confirmed with a wide range of parameter values.

[22] Intuitively, the cost of one unit of equilibrium consumption increases with increase in a person's uncertainty about their current of future earnings. Further, the presence of the LNIS may increase the positive skew of the distribution of equilibrium consumption price (or equilibrium state price density), and this change will affect existing frameworks for asset pricing and risk management.

$$K(r) = \left\{ \frac{\gamma - 1}{\gamma} \left(r + \frac{\gamma (\sigma^I)^2}{2} \right) + \frac{\beta}{\gamma} \right\}^{-\gamma}.$$

Proof. See an online appendix. **Q.E.D.**

Our equilibrium results with the LNIS on the risk-free interest rate sit somewhat easily with the key role of the savings inequality in shaping the wealth concentration among the rich (Piketty, 2014; Acemoglu and Robison, 2015). In the existing equilibrium literature without the LNIS, the equilibrium difference of $\beta - r$ is large, so higher-wealth individuals have a tendency to consume more and save less. This is because their financial wealth grows at the rate of β, whereas their savings grow at the rate of r. When financial wealth is disproportionately concentrated on the top of the wealth distribution, i.e., with the unequally distributed wealth, the rich dis-save in equilibrium. Thus, the low or nearly zero wedge of $\beta - r$ is the key to generate high savings rates for the wealthy.

How do we obtain such a low difference of $\beta - r$ in equilibrium? The possibility is to understand in equilibrium the effects of the LNIS on the interest rate. The representative agent facing the LNIS demands extra premium for her risk exposure to the LNIS, which corresponds to a decrease in β by the amount of the risk premium $\hat{\delta} - \delta$ depending upon Poisson arrival intensity δ of the LNIS. With the LNIS, the agent's demand for precautionary savings is sufficiently strong making her save at a high rate and lowering the equilibrium interest rate significantly, which will be further detailed in Section 5.

This effect of δ on interest rates is particularly relevant to today's low interest-rate situation, in that the LNIS has placed a heavy burden on the choice of equilibrium interest rate. This additional burden also leads to a reduction of interest rate. A wide range of

private and public insurance would greatly improve security of earnings in the long run. Thus, strategies and solutions focusing on the elimination of risk potentially catastrophic individual earnings losses as much as possible would help to reduce the amount of compensation for exposure to the LNIS, and might return the interest rate to its normal status during times of stable economy.

5 Quantitative Analysis

In this section, we perform quantitative analysis to illustrate the optimal strategies to attain consumption smoothing when the LNIS can occur.

Lower Bound of Wealth. In terms of empirical reality, we can relate the lower bound to a tightening of credit.

> In 2016, 20.8 percent of families were considered credit constrained – those who reported being denied credit in the past year, as well as those who did not apply for credit for fear of being denied in the past year (Survey of Consumer Finances, 2017).

Borrowing against human capital is constrained fully or partly. Thus, the extent to which credit is tightened, i.e., the level of lower bound of wealth becomes a real consideration.

Aggregate earnings are assumed to be given by a constant income stream. Specifically, the earnings are given by $\epsilon \equiv \xi I$ over the life cycle. Then, the lower bound $-L(t)$ with

$L(0) = L$ of wealth in (4) can be empirically plausible using the following relationship:[23]

$$L = \omega \frac{r + \nu + \delta k}{(r + \nu + \delta)(r + \nu)} \epsilon, \quad \text{for} \quad 0 \leq \omega < 1, \tag{15}$$

where ω represents the extent to which credit is tightened and $\nu > 0$ is the agent's constant mortality intensity when the time to death is distributed with an exponential distribution.[24] The utility related to death is normalized as zero.[25]

The relationship (15) shows that the value of human capital depreciates relative to that predicted by Friedman (1957). Without the LNIS (i.e., $\delta = 0$), the lower bound reduces to $\epsilon/(r + \nu)$ as ω approaches one (i.e., when credit tightening does not occur). With the LNIS (i.e., $\delta > 0$), the lower bound is always larger than $-\epsilon/(r + \nu)$, which is more credit tightened than without the LNIS.

5.1 Parameter Values

Financial Market. We consider only two assets in the financial market: a riskless bond and a risky stock. We choose equity premium, $\mu - r$, as 4% and risk-free rate, r, as 2%.

[23]The relationship shows an accurate reflection of human capital adjusted by the LNIS. The derivation details are available in an online appendix.

[24]The constant mortality rate assumption is made for parsimony of the model, helping explore horizon-dependent polices in the simplest possible economic environment. The derived model predictions are consistent with the typical life-cycle advice. A more realistic model would allow for a Gompertz force of mortality, which is quite relevant to the actuarial literature.

[25]On account of this normalization, we do not consider motive for bequest. The presence of bequest motive is expected to reinforce the negative impacts of the LNIS.

The stock volatility, σ, is assumed to be 20%.[26]

People Preferences. The coefficient of relative risk aversion, γ, is set to 2. We adopt the common value of 4% for the subjective discount rate, β. The mortality rate, ν, is fixed to 2%, i.e., the expected time to death is 50 years.

Aggregate Earnings. When a jump-type LNIS occurs, aggregate earnings decrease from ϵ to $k\epsilon$, $k \in (0, 1)$. For prototype example, we let the LNIS be associated with the risk of forced unemployment; this risk can be also a leading cause of the credit tightening formulated by (15).[27]

The Poission intensity δ can be thought of as an empirically plausible small annual probability of forced unemployment. The intensity δ is matched up with the mean 7% of the actual unemployment rates described in U.S. states monthly rankings seasonally adjusted Aug. 2013 (Source: Bureau of Labor Statistics). The forced unemployment then occurs 14-year from now on average. The recovery parameter k is set to 20%. In practice, U.S. households have been rescued by a safety net against forced unemployment, and

[26]Compared to the century-long sample (1891-1994) by Campbell (1999), the risk-free rate of 2% is reasonable, but the equity premium of 4% is somewhat conservative. In our general equilibrium analysis, the risk-free rate and equity premium can be determined to 2% and 4%, respectively, with the reasonable values of risk aversion.

[27]A key insight comes from economic recessions followed by human capital depreciation during long-term periods of unemployment. During the 2007-2009 Great Recession in the United States, many people have experienced the unprecedented largest reductions in their consumption and unemployment.

recover 20% of the income that they earned before unemployment (Carroll *et al.*, 2003).

5.2 Numerical Illustrations

In this section, we obtain numerical solutions and present graphical illustrations to provide some details to the discussion about the optimal strategies and general equilibrium quantities with the LNIS.

In subsequent analysis, we will fix one dimension by setting $\omega = 0$, which considers the most extreme credit-tightened scenario. We then investigate the effects of the other dimension by varying δ. For the details about the effects of credit tightening, refer to an online appendix.

Precautionary Savings. The optimal consumption amount would fall more in the presence of the LNIS than in its absence (Figure 1).

In the absence of the LNIS ($\delta = 0$), similar to Bewley (1977) and Campbell (1987), we could confirm the amount of precautionary savings decreases in financial wealth w (not reported here). These results imply that people can self-insure by saving financial resources from a constant income stream when financial markets are stable and by expending such resources on the management of adverse economic shocks in the markets.

However, in the presence of the LNIS ($\delta > 0$), the amount of precautionary savings (quantified in Section 3) does not decrease as wealth increases (Figure 2). Rather, this amount is an increasing and concave function of wealth. People would be better off keeping

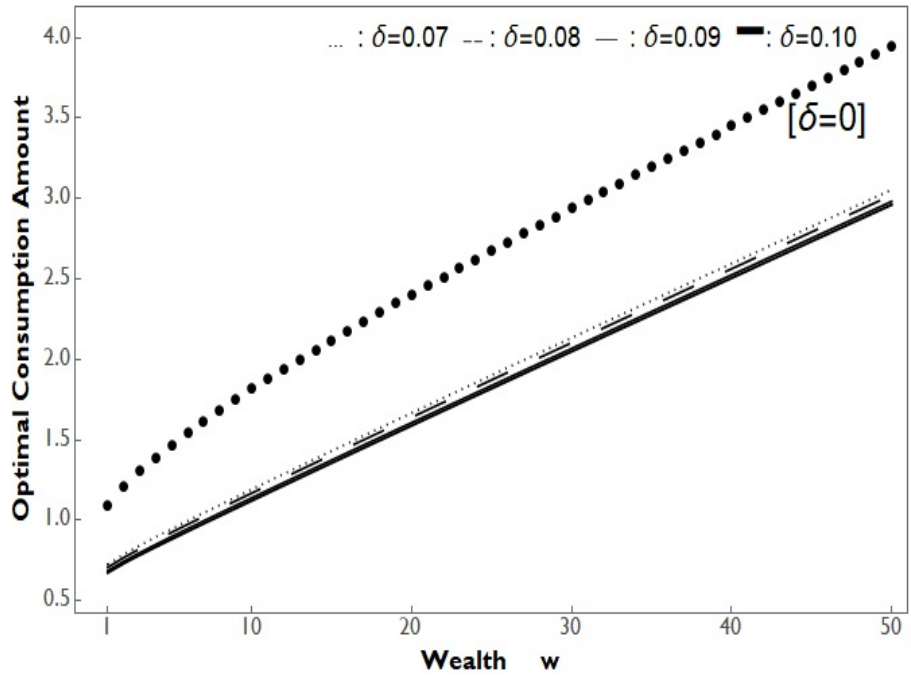


Figure 1: **Optimal consumption amount.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate). Note: The optimal consumption amount would fall more in the presence of the LNIS than in its absence.

accumulating extra wealth for precautionary reasons, and use this extra wealth as a buffer against the LNIS. However, an unexpected, exogenous, and permanent jump-type LNIS is much harder to buffer than adverse diffusive-type income shocks, so readiness for the LNIS requires a large amount of savings.

The percentage precautionary savings is defined as the difference of precautionary savings with and without the LNIS, divided by the optimal consumption amount of Merton (1969, 1971); i.e., the target amount of consumption. Interestingly, this percentage shows a hump-shaped relationship to wealth. Contrary to the predictions of Bewley (1977) and Campbell (1987), the LNIS should affect significantly the savings rate of rich people. Em-

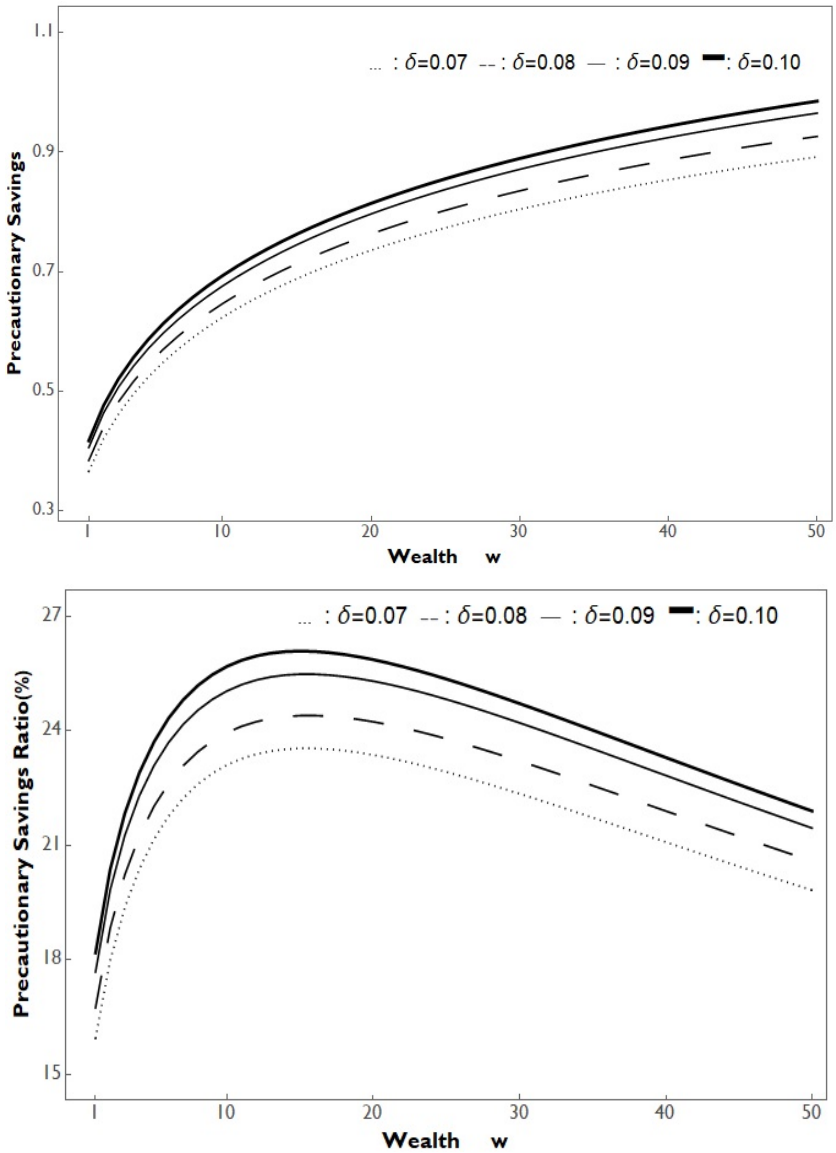


Figure 2: **Precautionary savings amount and ratio.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate). Note: Contrary to the case in the absence of the LNIS ($\delta = 0$), in its presence ($\delta > 0$) the amount of precautionary savings does not decrease as wealth increases. Rather, this amount is an increasing and concave function of wealth.

pirically, the average ratio of wealth-to-income between 1952 and 2016 in the U.S. was 6.6. Since labor income is normalized as one in our paper, high-wealth people can be regarded as those having wealth more than 6.6. We find that the possibility of the LNIS increases the precautionary savings as wealth increases, thereby explaining high savings rates of the wealthy rather than negative savings rates predicted by Bewley (1977) and Campbell (1987). Overall, high-wealth people have higher proportional savings than low-wealth people.

The recent studies offer many alternative explanations for the upward-sloping part of individual savings profile: a positive relation between savings rates and income (Dynan *et al.*, 2004), entrepreneurship purposes for entering and expanding business (Buera, 2009), out-of-pocket medical expenses patterns (De Nardi *et al.*, 2010), the mix of bequests and human capital, entrepreneurship, and medical-expense risk (De Nardi and Fella, 2017). The explanation we provide here, while different, would be regarded as complementary to these.

The income-to-wealth-ratio is inversely related to individual wealth. The income is a major staple of the relatively low-wealth people and they should concern themselves with diversifying the negative effects of the LNIS, thereby continuing being able to afford what they can currently afford by saving less. Relative to the low-wealth people, the income is a smaller staple of the high-wealth people, so they have greater tolerance for risk than the low-wealth people. Hence, the high-wealth people would rather not be concerned with diversification. They can afford to pay further for economic preparedness and emergency savings, thereby saving more to meet the required precautionary savings for consumption

smoothing. They would, thus, be well-prepared for unexpected hardship caused by the LNIS.

The downward-sloping part of individual savings profile can be understood within the traditional life-cycle framework. In the context of the life-cycle hypothesis, the savings should have the downward-sloping part at the high end of wealth (Jappelli, 1999; Deaton and Paxson, 2000; Attanasio and Szekely, 2000; Cocco *et al.*, 2005; Benzoni *et al.*, 2007).[28] For very-high-wealth people, the LNIS does not have sufficient power to affect as their enough wealth absorbs the negative effects of the LNIS. In this case, income with the LNIS takes on riskless savings-like feature. As a result, the very-high-wealth people start to reduce their savings. This creates the downward-sloping part of the savings profile.

Human Capital Value. Neglecting the LNIS can be costly to individuals who aim to attain their consumption smoothing in terms of human capital aspect. We measure the value of human capital as the marginal rate of substitution between income and financial wealth. That is, the human capital value can be regarded as the individual's subjective marginal value of her income.

Similar to the relation of (15), the actuarial fair value (AFV) of future income dis-

[28]Empirically, the downward-sloping part has been observed by Jappelli (1999) using data from Italy, by Deaton and Paxson (2000) using data from Taiwan and Thailand, and by Attanasio and Szekely (2000) using data from East Asia and Latin America. These empirical observations have been justified by theoretical life-cycle models such as Cocco *et al.* (2005) and Benzoni *et al.* (2007).

counted at the risk-free interest rate is[29]

$$AFV = \frac{r + \nu + \delta k}{(r + \nu + \delta)(r + \nu)}\epsilon,$$

which is the present value of income. We use AFV as a benchmark against the effects of the LNIS on the human capital value.

Definition 5.1 *Let $V(w, I; \delta)$ be the value function given in (3) with the Poisson intensity δ. Then, the value of human capital is defined as the marginal rate of substitution between income and financial wealth, i.e.,*

$$\frac{\partial V(w, I; \delta)}{\partial I} \Big/ \frac{\partial V(w, I; \delta)}{\partial w}.$$

The LNIS dramatically reduces the value of human capital, regardless of levels of financial wealth (Figure 3). As expected, the level of human capital decreases as the chance of income shock (δ) increases. The wage would thus decrease with the LNIS.

Interestingly, we could see an increasing and concave trend of the human capital value with respect to wealth, and the concavity strengthens as wealth decreases. This trend would be especially problematic for the poor people because they have very little residual income to save with the LNIS and therefore may be ill-prepared for the situations in its aftermath.

The increasing and concave human capital value in wealth would have relevance to the wealth concentration among the wealthy. While the rich theoretically do not save (Bewley,

[29] In the actuarial literature, it is common to use the risk-free interest rate for actuarial calculation purposes.

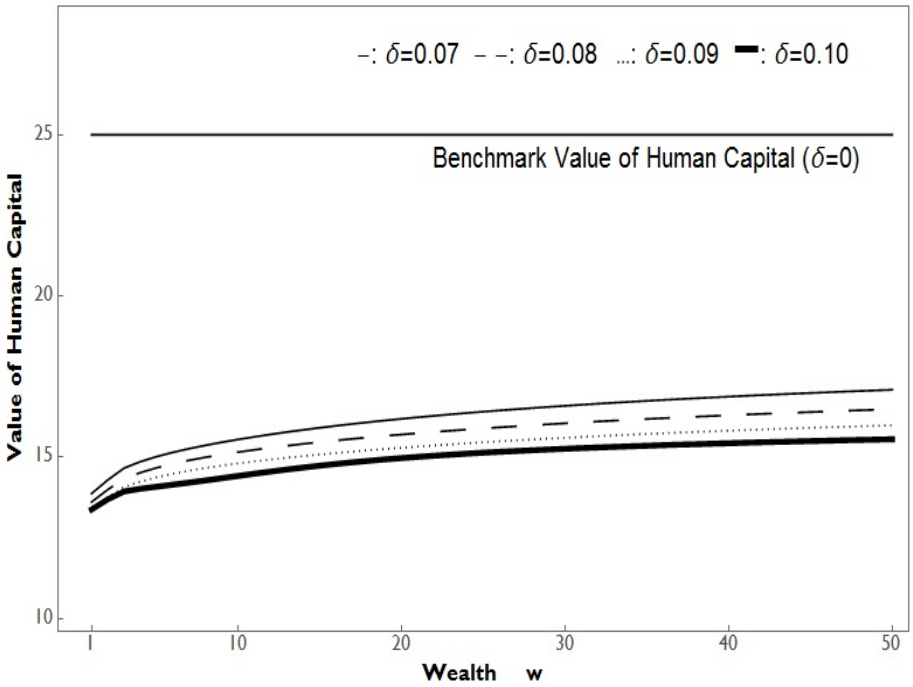
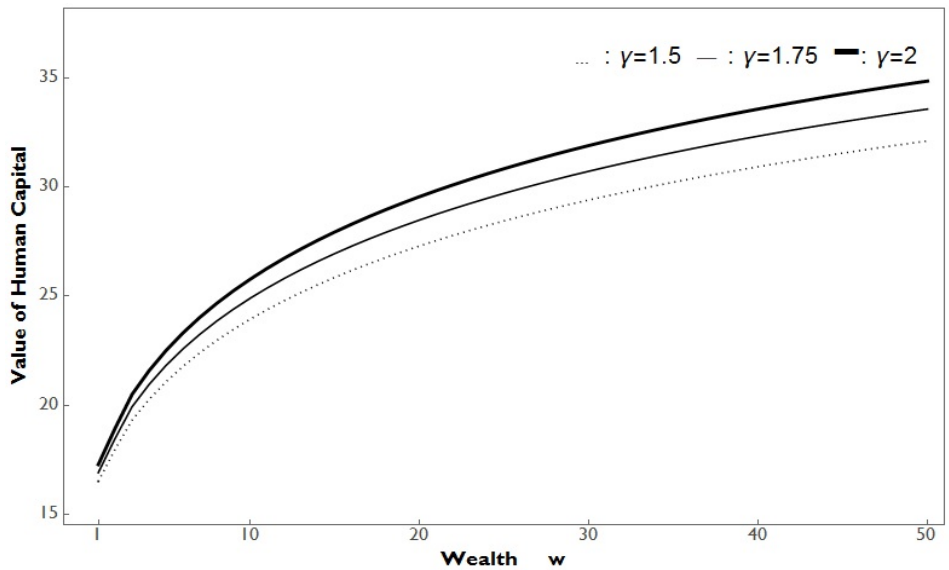
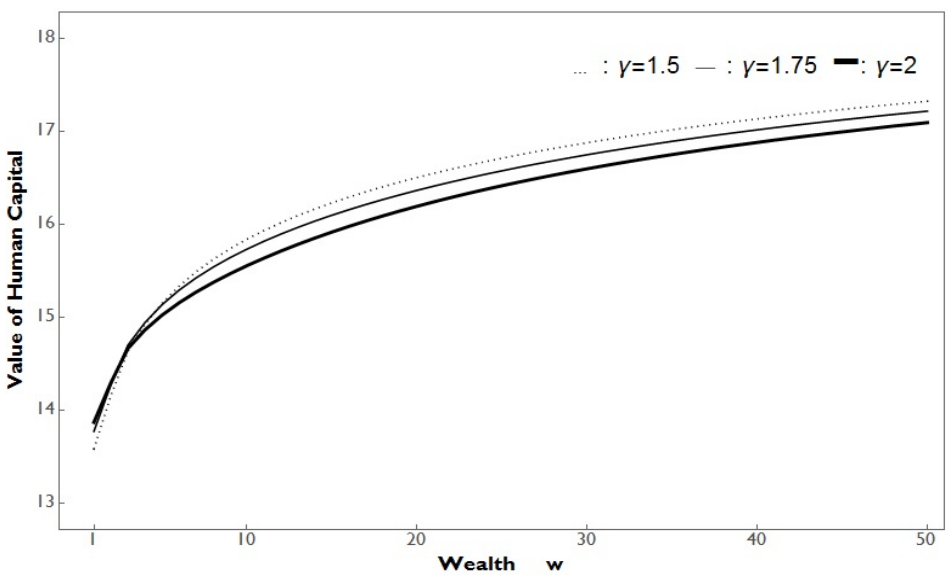


Figure 3: **Human capital value**. Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate). Note: The LNIS dramatically reduces the value of human capital, regardless of levels of financial wealth. The human capital value has an increasing and concave trend and the concavity strengthens as wealth decreases.



(a) $\delta = 0$



(b) $\delta = 0.07$

Figure 4: **The sensitivity of human capital value with respect to changes of risk aversion**. Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate). Note: An increase of risk aversion increases the human capital value in the absence of the LNIS. In contrast, in the presence of the LNIS, an increase of risk aversion decreases the human capital value.

1977; Campbell, 1987), they empirically have strong incentives to save and show the wealth concentration (Kaplan and Rauh, 2013).[30] In terms of the human capital aspect, the top wealth-rich are high-income households and thus, their relatively high exposure of labor income to the LNIS is a likely contributor to their high precautionary savings rates for the preparation after the LNIS, which is consistent with the recent observations in the exposure of labor income to aggregate fluctuations (Parker and Vissing-Jørgensen, 2009; Guvenen *et al.*, 2017).

The effects of the LNIS outweigh the effects of risk aversion (Figure 4). An increase of risk aversion increases the human capital value in the absence of the LNIS. Intuitively, as an individual's risk aversion increases, she becomes increasingly likely to increase her investment in (seemingly) riskless human capital than in the risky assets (increasing her taking on market risk). In contrast, in the presence of the LNIS, an increase of risk aversion decreases the human capital value, so it no longer serves as a substitute of riskless assets, rather it resembles like a defaultable risky asset. Of course, the concern about the LNIS increases as risk aversion increases.

Equilibrium Interest Rate. In the absence of the LNIS ($\delta = 0$), the equilibrium interest rate is 6.55%, but in the presence of the LNIS ($\delta > 0$), it drops significantly (Figure 5). This relationship implies the important discontinuity and dramatic change in the interest rate even when δ is small. For instance, the equilibrium interest rate decreases 45.04%

[30]The wealthiest 400 Americans on the Forbes Magazine list own 1.5% of the total wealth in the US (Kaplan and Rauh, 2013).

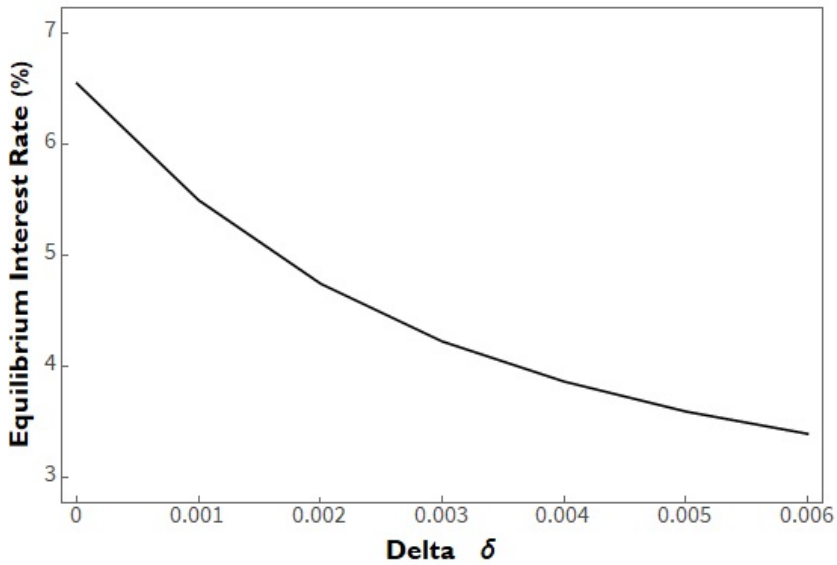


Figure 5: **Equilibrium interest rates**. Parameter Values: $\beta = 4\%$ (subjective discount rate), $\gamma = 2$ (risk aversion), $\mu^I = 1.52\%$ (expected consumption growth rate), $\sigma^I = 4.06\%$ (volatility of consumption growth rate), $w = 1$ (initial wealth), $I = 1$ (aggregate output), $\xi = 0.5$ (fraction constituting aggregate earnings), and $k = 20\%$ (recovery rate). For the expected consumption growth rate and volatility of consumption growth rate, μ^I and σ^I, we have used the Robert J. Shiller's real monthly dividend data from 1926 to 2016 in "Irrational Exuberance" published by Princeton University Press. Note: In the absence of the LNIS ($\delta = 0$), the equilibrium interest rate is 6.55%, but in the presence of the LNIS ($\delta > 0$), it drops significantly, implying the important discontinuity and dramatic change in the interest rate even when δ is small.

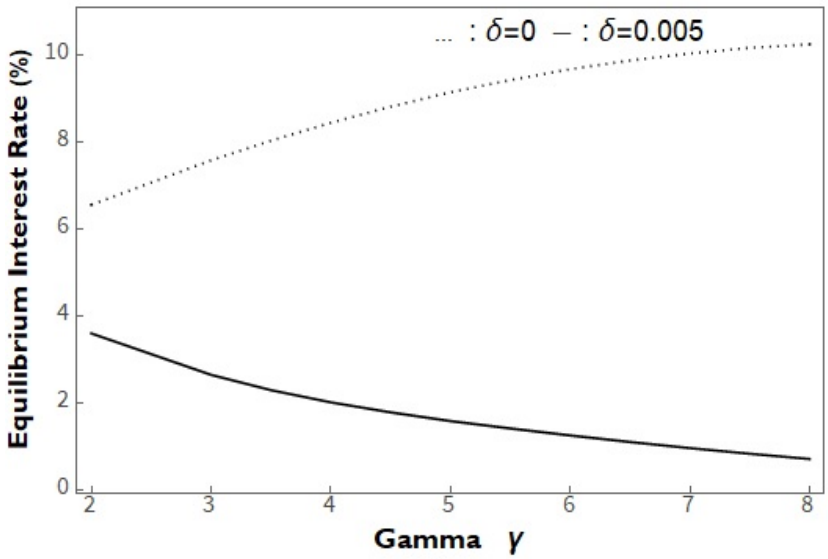


Figure 6: **Equilibrium interest rates**. Parameter Values: $\beta = 4\%$ (subjective discount rate), $\gamma = 2$ (risk aversion), $\mu^I = 1.52\%$ (expected consumption growth rate), $\sigma^I = 4.06\%$ (volatility of consumption growth rate), $w = 1$ (initial wealth), $I = 1$ (aggregate output), $\xi = 0.5$ (fraction constituting aggregate earnings), and $k = 20\%$ (recovery rate). Note: When $\delta > 0$, high values of γ no longer counterfactually generage a high risk-free interest rate, so the risk-free rate puzzle (Weil, 1990) is avoided. Rather, an increase in risk aversion can lead to a decrease in risk-free rate in the presence of the LNIS.

(i.e., to 3.60%) as δ increases from 0 to 0.5%.

Risk aversion γ also affects the equilibrium risk-free interest rates (Figure 6). The presence of the LNIS drives down the risk-free interest rate by stimulating the precautionary savings mechanism. When $\delta > 0$, high values of γ no longer counterfactually generate a high risk-free interest rate, so the risk-free rate puzzle (Weil, 1990) is avoided. Rather, an increase in risk aversion can lead to a decrease in risk-free rate in the presence of the LNIS. The precautionary savings motive maintains the risk-free rate low. This result can be understood in the context of the rate disaster hypothesis (Rietz, 1988).[31] Consistent

[31]The rate disaster hypothesis arguably states that the slim chance of rare disasters (e.g., economic

Estimated consumption and return parameters	1891-1994	1947.2-1996.3
Expected consumption growth rate μ^I	1.74%	1.91%
Consumption volatility σ^I	3.26%	1.08%
Stock Volatility σ	18.53%	15.22%
Risk-free rate r	1.96%	0.79%
Equity premium $\mu - r$	6.26%	7.85%

Table 1: The table reports the annualized parameter values for consumption and return by Campbell (1999) for the century-long sample (1891-1994) and the postwar sample (1947.2-1996.3).

with this hypothesis, the possibility of the LNIS can account for high risk premium on bonds.

Matching Equity Premium and Risk-Free Rate. Using the LNIS, our equilibrium results help match the observed risk-free rate and equity premium. We have tried to match our model with the data taken from Campbell (1999): the century-long sample from 1891 to 1994 and the postwar sample from 1947.2 to 1996.3 (Table 1). The presence of the LNIS dramatically improves the model's ability to match asset prices (Figure 7). For a fixed subjective discount rate $\beta = 0.04$, our equilibrium risk-free rate and equity premium with the LNIS (when $\delta = 0.005$) match the century-long data much better than

crisis or war) can dominate the determinaiton of asset risk premia. The seminal work of Rietz (1988), Barro (2006) and Gabaix (2012) have established different versions of the rate disaster hypothesis, thereby explaining empirical regularities, such as the equity premium puzzle and the risk-free rate puzzle.

the standard asset pricing model without the LNIS (when $\delta = 0$) does. Not only the standard asset pricing framework requires a risk aversion coefficient more than the upper bound 10 (Mehra and Prescott, 1985) to match the equity premium of 6.64%, but it also generates an excessively high risk-free rate, which gives rise to the risk-free rate puzzle. While the equilibrium quantities generated by ours with the LNIS are much closer to the century-long sample data reporting the equity premium of 6.64% and the risk-free rate of 1.64% with reasonable coefficients for risk aversion less than the upper bound 10.

The postwar sample is hard to match, because it is characterized by low consumption and high stock volatilities (not reported). The standard asset pricing model is particularly problematic because it requires a very high risk aversion coefficient to match the equity premium of 7.89%. At the same time, the risk-free rate puzzle occurs as well due to such a high coefficient for risk aversion. Our model with the LNIS is found to generate the equity premium of 1.64% with the upper bound 10 of risk aversion, which is, of course, much lower than the historical premium of 7.85%. However, our model partly explains the risk-free rate of 1.35% with the risk aversion of 10, which is higher than the historical rate of 0.79%.

The intuitive interpretation of our improved ability to match with the observed asset prices is as follows. Individuals are inclined to be concerned with the observed equity premium, because it seems too good to be true in the event of incoming catastrophic earnings shock. The LNIS faced by individuals cause them likely to reduce consumption to secure extra reserves as a precaution or to finance consumption needs by using their savings. Such a conservative decision encourages reduction in equity investment, so the

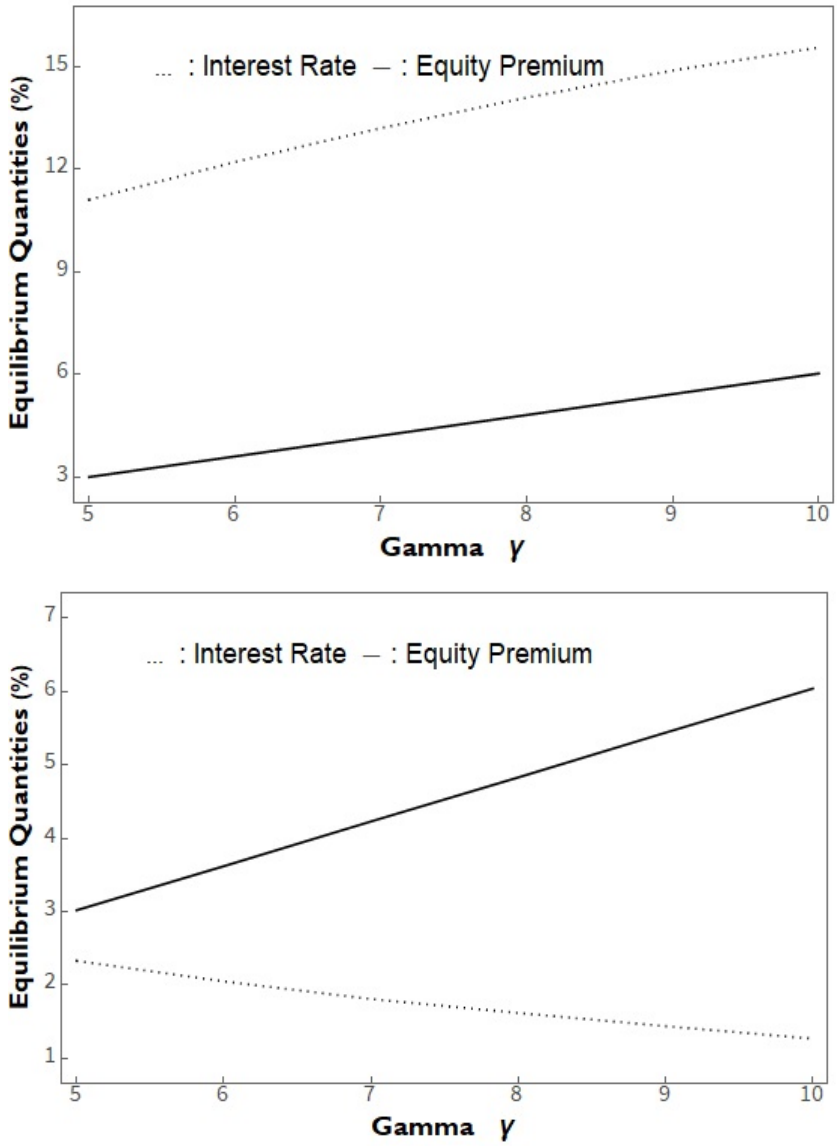


Figure 7: **Equilibrium quantities**. The figures show the equilibrium risk-free rate and equity premium with risk aversion. For the parameter values of expected consumption growth rate and consumption volatility, we have used the century-long sample (1891-1994) data in Table 1. The top figure and bottom figure show the equilibrium quantities when $\delta = 0$ (with no income shock) and $\delta = 0.005$ (with income shock). The subjective discount rate β is fixed to 0.04. Note: For a fixed subjective discount rate $\beta = 0.04$, the equilibrium risk-free rate and equity premium with the LNIS (when $\delta = 0.005$) match the century-long data (the equity premium: 6.64% and the risk-free rate: 1.64%) much better than the standard asset pricing model with the LNIS (when $\delta = 0$) does.

equity premium increases and the risk-free rate decreases.

6 Conclusion

We have developed an analytically tractable framework that generalizes Friedman's permanent income hypothesis (PIH) by considering the possibility of a large, negative income shock (LNIS). The generalized PIH with the optimal consumption and investment strategies helps attain consumption smoothing when the LNIS occurs. We have quantified precautionary savings and calculated the required amount of savings for consumption smoothing. We find an increasing and concave trend of the savings with respect to wealth and that high-wealth people could save more than low-wealth people. Obviously, such savings pattern would also alter existing portfolio strategy, suggesting wealth- and age-dependent investment rule.

We have provided a general equilibrium analysis based on the Lucas-type pure exchange economy. The equilibrium interest rate with the LNIS is obtained in closed-form. We find that the substantial amount of extra precautionary savings for consumption smoothing, driven by high-wealth people, should be reflected in the equilibrium asset pricing. We show that the equilibrium interest rate would therefore fall dramatically even when the chance of the LNIS is small; this result partly explains today's low interest rates. Finally, our equilibrium model's ability to match asset prices observed in the data is greatly improved with the LNIS.

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Online Appendix for "A Generalization of Friedman's Permanent Income Hypothesis with a Large, Negative Income Shock"

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1 A Two-period Model

We start with a simple two-period model with a jump-type income shock for having a good motivation. We consider a representative economic agent who is searching for her optimal consumption and investment strategies over the two periods: period 0 and period 1. The agent dies at the end of period 1 and the probability of her survival until time 1 is δ_1. The objective of the agent is to maximize the following utility function

$$v(c_0) + \delta_2 E v(c_1),$$

where v is a strictly increasing, strictly concave real-valued function defined on the set of positive real numbers, $0 < \delta_2 < 1$ is the subjective discount factor, and E denotes the expectation taken at time 0.

There are two tradable financial assets: a riskless bond and a risky stock. The riskless bond pays 1 at period 1 and its price is $\frac{1}{R}$ at period 0, where $R > 0$ is the risk-free interest rate. The price of a share of the risky stock is 1 at period 0 and can be u and d ($u > R > d > 0$) with probabilities π_u and $\pi_d = 1 - \pi_u$, respectively, at period 1. The agent obtains aggregate earnings at the rate of ϵ in each period. There is a jump shock in her earnings that would cause a significant downward jump in earnings from ϵ to 0 at period 1 with the probability of p. The probability distributions of the agent's mortality, the stock price, and the jump shock are assumed to be independent.

The budget constraint during period 1 is described as the following: for $i \in \{u, d\}$,

$$W_{1i} = \begin{cases} Rw_0^B + iw_0^S + \epsilon, & \text{if the income shock does not occur,} \\ Rw_0^B + iw_0^S, & \text{if the income shock occurs,} \end{cases}$$

where w_0^B is the dollar amount of savings invested in the riskless bond during period 0, and w_0^S is the

dollar amount of savings invested in the risky stock during period 0.

The optimal consumption strategy c_{1i} at period 1 for $i \in \{u, d\}$ is to consume all of wealth W_{1i} available at period 1 i.e., $c_{1i} = W_{1i}$. The budget constraint during period 0 is given by

$$W_0 + \epsilon = c_0 + w_0^B + w_0^S,$$

where c_0 is the optimal consumption strategy at period 0.

The agent's optimization problem at period 0 is formulated by the following value function:

$$\begin{aligned}
&\max_{(w_0^B, w_0^S)} \Big[v\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) + \delta_2 E v\Big(W_1\Big)\Big] \\
&= \max_{(w_0^B, w_0^S)} \Big[v\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) \\
&\qquad + \delta(1-p)\Big\{\pi_u v\Big(Rw_0^B + uw_0^S + \epsilon\Big) + \pi_d v\Big(Rw_0^B + dw_0^S + \epsilon\Big)\Big\} \\
&\qquad + \delta p\Big\{\pi_u v\Big(Rw_0^B + uw_0^S\Big) + \pi_d v\Big(Rw_0^B + dw_0^S\Big)\Big\}\Big],
\end{aligned}$$

where $\delta \equiv \delta_1 \delta_2$. The first-order conditions for w_0^B and w_0^S are given by

$$\begin{aligned}
&v'\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) \\
&= (1-p)\delta R\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S + \epsilon\Big) + \pi_d v'\Big(Rw_0^B + dw_0^S + \epsilon\Big)\Big\} \\
&+ p\delta R\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S\Big) + \pi_d v'\Big(Rw_0^B + dw_0^S\Big)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&v'\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) \\
&= (1-p)\delta\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S + \epsilon\Big)u + \pi_d v'\Big(Rw_0^B + dw_0^S + \epsilon\Big)d\Big\} \\
&+ p\delta\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S\Big)u + \pi_d v'\Big(Rw_0^B + dw_0^S\Big)d\Big\},
\end{aligned}$$

respectively.

We assume the simplest possible utility function: v is quadratic and it is given by

$$v(c) = c - \gamma c^2,$$

where γ is a positive constant. Then the first-order conditions become linear equations and we can

derive in closed-form the optimal strategies as the following:

$$
\begin{aligned}
w_0^B = \Big[&\delta\{-1+\gamma(W_0+\epsilon)\}\{u\pi_u(u-R)-d\pi_d(R-d)\} \\
&+\delta\{(1-p)\gamma\epsilon-1\}\{(u\pi_u+d\pi_d)-R(1-2\delta ud\pi_u\pi_d)\}\Big] \\
&\Big/\gamma\Big[\{1+\delta R^2\}\{1+\delta(u^2\pi_u+d^2\pi_d)\}-\{1+\delta R(u\pi_u+d\pi_d)\}^2\Big],
\end{aligned}
$$

$$
\begin{aligned}
w_0^S = \Big[&\delta R\{-1+\gamma(W_0+\epsilon)\}(u\pi_u+d\pi_d-R) \\
&+\delta\{(1-p)\gamma\epsilon-1\}\{(u\pi_u+d\pi_d)-R\}\Big] \\
&\Big/\gamma\Big[\{1+\delta R(u\pi_u+d\pi_d)\}^2-\{1+\delta R^2\}\{1+\delta(u^2\pi_u+d^2\pi_d)\}\Big].
\end{aligned}
$$

Notice that the premium term $(u\pi_u + d\pi_d) - R$ on the risky stock can be reasonably assumed to be positive. Then,

$$\frac{w_0^S}{\partial p} < 0.$$

This arguably states that the income shock reduces the dollar amount of savings invested in the risky stock. However, we fully acknowledge that this result might depend upon the quadratic utility function and two-period assumptions.

As far as we know, the first-order conditions when deriving the optimal strategies in an incomplete market are highly non-linear. Those conditions are hardly possible to be solved analytically even for the well-known cases such as the constant absolute risk aversion or the constant relative risk aversion utility functions. If we extend the framework to multi-period settings, the situation is even worsen.

Starting from the next section, we will develop a very tractable continuous-time model in which all the optimal strategies are derived in closed-form and the equilibrium asset prices are in closed-form as well.

2 A Continuous-Time Model

2.1 The Building Blocks

The Benchmark Problem. In the context of the consumption utility maximization framework, the representative agent's consumption-savings model can be formulated by

$$V^B(w) \equiv \sup_c E\Big[\int_0^\infty e^{-\beta t} U\big(c(t)\big)\Big], \tag{1}$$

subject to the following wealth process of the agent:

$$dW(t) = \{rW(t) - c(t) + I\}dt, \;\; W(0) = w > -I/r,$$

and

$$W(t) > -\frac{I}{r}, \;\; \text{for all } t \geq 0,$$

where r is the risk-free interest rate and $I > 0$ stands for reduced-form aggregate earnings.

Throughout the paper, we consider the following constant relative risk aversion (CRRA) utility function:

$$U\big(c(t)\big) = \frac{c(t)^{1-\gamma}}{1-\gamma},$$

where $\gamma > 0$ is the constant coefficient for an agent's relative risk aversion. We derive in closed-form the solution of the optimization problem of (1):

$$V^B(w) = \frac{1}{r}\frac{(rw+I)^{1-\gamma}}{1-\gamma}, \;\; w > -I/r.$$

Accordingly, we also derive in closed-form the optimal consumption-savings strategy:

$$c(t) = rw + I. \tag{2}$$

The Permanent Income Hypothesis. We revisit the classic permanent income hypothesis of Friedman (1957) and Modigliani and Brumberg (1954) using the derived optimal strategy by (2). A little rearrangement of the optimal consumption strategy (2) shows that the ratio of consumption to total

available financial resources is constant. More precisely,

$$\frac{c(t)}{w + I/r} = r,$$

where the denominator $w + I/r$ denotes total wealth comprised of financial wealth w and human capital I/r, the present value of future aggregate earnings discounted at the risk-free rate (Friedman, 1957; Hall, 1978). Note that the marginal consumption with respect to financial wealth w is always constant, predicting consumption smoothing. Intuitively, human capital strongly holds up total available financial resources and thus, people are able to attain the constant standard of living through consumption smoothing as the permanent income hypothesis predicts.

In reality, people can choose to finance very large expenditures such as buying a car, buying a house, sending their children to college, and so on, rather than directly relying on cash. Here, human capital plays a key role in smoothing consumption as people can shift their earning power of labor from high-income periods to low-income periods of life. In high-income periods people can finance consumption needs by borrowing against their human capital, keeping cash in banks. In low-income periods people can use their savings for the most satisfying standard of living.

The Optimal Consumption and Investment Problem in an Incomplete Market. We recollect the representative agent's optimal consumption and investment problem in an incomplete market specified by the following stochastic optimization problem:

$$V(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K \frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \tag{3}$$

with the following dynamic wealth constraint:

$$dW(t) = \{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\}dt + \pi(t)^\top\sigma^\top dZ(t),\ \ W(0) = w > -\xi I/\beta_1, \tag{4}$$

$$W(t) > -\frac{\xi I(t)}{\beta_1},\ \ \text{for all } t \geq 0,$$

where

$$K = A^{-\gamma},\ A = \frac{\gamma - 1}{\gamma}\Big(r + \frac{||\theta||^2}{2\gamma}\Big) + \frac{\beta}{\gamma},\ \beta_1 = r - \mu^I + (\sigma^I)^\top\theta,\ \theta = (\sigma^\top)^{-1}(\mu - r\mathbf{1}).$$

The technical details behind the derivation of the optimization problem (3) are given as follows. We have assumed that the large, negative income shock is driven by a Poisson shock. On account of such an assumption, the problem is the same as the traditional Merton's (1969, 1971) case after the arrival of the Poisson shock. That is, the agent's problem after the Poisson shock is given by

$$V^A(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt\Big], \tag{5}$$

subject to

$$dW(t) = \{rW(t) - c(t) + k\xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\}dt + \pi(t)^\top\sigma^\top dZ(t),\ \ W(0) = w > -\xi I/\beta_1, \tag{6}$$

$$W(t) > -\frac{\xi I(t)}{\beta_1},\ \ \text{for all } t \geq 0.$$

Note that the agent undergone the significant reduction in her aggregate earnings from $\xi I(t)$ to $k\xi I(t)$, $k \in (0, 1)$, as identified in (6). Ultimately, the problem (5) belongs to the conventional utility-maximizing framework in which it is optimal to maximize the consumption of goods and services over the life cycle. Following Merton (1969, 1971), the problem is solved in closed-form:

$$V^A(w, I) = K\frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}. \tag{7}$$

Before integrating out the Poisson intensity δ, the original problem should be given by

$$V(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\tau e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}V^A(w, I)\Big],$$

where τ represents the arrival of the Poisson shock. After integrating out the Poisson intensity δ, the problem stated above becomes the problem (3).

2.2 The Solution

Problem Reformulation. Following Karatzas *et al.* (1991), we come up with a fictitious asset, $S^f(t)$, and its price dynamics are given by

$$dS^f(t) = \mu^f S^f(t)dt + \sigma^f S^f(t)dM(t),$$

where μ^f and σ^f are to be determined under the minimal local martingale measure and $M(t)$ is a compensated martingale process related to a Poisson shock. Specifically, the dynamics of $M(t)$ follow

$$dM(t) = -\delta dt + dN(t),$$

where $N(t)$ represents the Poisson shock. By defining the market price of the income shock (driven by the Poisson) shock as

$$\hat{\delta} \equiv \delta + \frac{\mu^f - r}{\sigma^f},$$

which will be called the income-shock-adjusted intensity. The state price densities are then characterized by

$$\xi^{\hat{\delta}}(t) \equiv \exp\Big\{ \ln\Big(\frac{\hat{\delta}}{\delta}\Big)\mathbf{1}_{\{\tau \le t\}} - (\hat{\delta} - \delta)t\Big\} H(t),$$

where **1** is an indicator function that gives 1 if the Poisson shock occurs at time t and 0 otherwise, $H(t)$ is the standard state price density in complete markets under no arbitrage and its dynamics are given by

$$dH(t) = -H(t)\{rdt + \theta^\top dZ(t)\}, \;\; H(0) = 1.$$

We provide a lemma to convert the dynamic wealth constraint in (4) into the static wealth constraint as follows.

Lemma 2.1. *The dynamic wealth constraint in (4) can be converted into the following static wealth constraint:*

$$E\Big[\int_0^\infty e^{-\hat{\delta}t} H(t)\Big(c(t) - \xi I(t) + \hat{\delta} W(t)\Big)dt\Big] \le w. \tag{8}$$

Proof. See 3. **Q.E.D.**

With the help of Lemma 3.1, we are able to convert the original stochastic optimization problem (3) into the following static optimization problem:

$$\sup_{(c,W)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K \frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \tag{9}$$

subject to

$$E\Big[\int_0^\infty e^{-\hat{\delta}t} H(t)\Big(c(t) - \xi I(t) + \hat{\delta} W(t)\Big)dt\Big] \le w.$$

We introduce another lemma to reformulate the problem (9) one more.

Lemma 2.2. *The static optimization problem of (9) can be reformulated as*

$$\begin{aligned}
&\inf_{(\lambda,\hat{\delta})}\{J^{\hat{\delta}}(\lambda,I)+\lambda w\}\\
&=\inf_{\lambda}\{\inf_{\hat{\delta}} J^{\hat{\delta}}(\lambda,I)+\lambda w\}\\
&\equiv\inf_{\lambda}\{J(\lambda,I)+\lambda w\},
\end{aligned} \tag{10}$$

where the indirect value function $J^{\hat{\delta}}(\lambda,I)$ is given by

$$\begin{aligned}
J^{\hat{\delta}}(\lambda,I)&=(\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^\infty e^{-(\beta_2+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(1+(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big)\Gamma^{\hat{\delta}}(t)^{1-1/\gamma}+\Big(1+\frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big]\\
&\equiv(\xi I)^{1-\gamma}\varphi_{\hat{\delta}}(z)
\end{aligned} \tag{11}$$

with $z=\lambda(\xi I)^\gamma$, where $\tilde{E}$ is the expectation under the new probability measure defined as

$$\tilde{P}(A)\equiv\int_A \exp\Big(-\frac{1}{2}(1-\gamma)^2||\sigma^I||^2(t,\omega)+(1-\gamma)(\sigma^I)^\top Z(t,\omega)\Big)dP(\omega)\ \ \textit{for all}\ \ A\in\mathcal{F}$$

with the new Brownian motion process $\tilde{Z}$ given by

$$\tilde{Z}(t)=-(1-\gamma)\sigma^I dt+Z(t),$$

$\Gamma^{\hat{\delta}}(t)$ *is a new state variable defined by*

$$\Gamma^{\hat{\delta}}(t)\equiv\lambda e^{(\beta+\delta-\hat{\delta})t}H(t)(\xi I(t))^\gamma,$$

and its dynamics are given by

$$d\Gamma^{\hat{\delta}}(t)=\Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}}-\beta_2)dt+\beta_3 d\tilde{Z}(t)\},$$

$$\begin{aligned}
&\beta_1^{\hat{\delta}}=r-\mu^I+(\sigma^I)^\top\theta+\hat{\delta}-\delta,\\
&\beta_2=\beta-\mu^I(1-\gamma)+\frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,\\
&\beta_3=\gamma(\sigma^I)^\top-\theta^\top.
\end{aligned}$$

Proof. See 3. **Q.E.D.**

Minimal Local Martingale Measure. In order to solve the reformulated problem (10), the income-shock-adjusted intensity $\hat{\delta}$ should be determined appropriately to find the minimal local martingale measure, guaranteeing the uniqueness of the state price density. Using the dynamic programming approach of Bensoussan *et al.* (2016), we determine $\hat{\delta}$ uniquely.

Lemma 2.3. *The income-shock-adjusted intensity $\hat{\delta}$ is determined uniquely by*

$$\hat{\delta} = \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{z}, \tag{12}$$

where $G(z) = -\varphi'_\delta(z) + 1/\beta_1$ solves to the following non-linear differential equation:

$$\begin{aligned} -\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) \\ + \beta_1 G(z) + \delta K \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \ \ 0 < z < \overline{z}. \end{aligned} \tag{13}$$

with the boundary conditions

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \ \ \textit{and} \ \ G'(\overline{z}) = 0.$$

Proof. See 3. **Q.E.D.**

Now, it remains to solve the non-linear differential equation (13) to characterize the income-shock-adjusted intensity $\hat{\delta}$ explicitly.

Proposition 2.1. *A general solution of the differential equation of (13) is given by*

$$\begin{aligned} G(z) =& \frac{1}{\hat{A} + \delta} z^{-1/\gamma} + B^*(\overline{z}; \delta) z^{-\alpha^*_\delta} \\ &+ \frac{2\delta K}{||\beta_3||^2 (\alpha_\delta - \alpha^*_\delta)(1-\gamma)} \Big[(\alpha_\delta - 1) z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2} \Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\ &+ (\alpha^*_\delta - 1) z^{-\alpha^*_\delta} \int_z^{\overline{z}} \mu^{\alpha^*_\delta - 2} \Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \Big], \end{aligned} \tag{14}$$

where $\hat{A}$ is given by

$$\hat{A} = \frac{\gamma - 1}{\gamma} \Big(\beta_1 + \frac{||\beta_3||^2}{2\gamma}\Big) + \frac{\beta_2}{\gamma},$$

$B^*(\overline{z};\delta)$ and $\overline{z}$ are the two constants to be determined by the boundary conditions:

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \ \ \textit{and} \ \ G'(z) = 0,$$

and $\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots to the following characteristic equation:

$$F(\alpha;\delta) \equiv -\frac{1}{2}||\beta_3||^2\alpha(\alpha-1) + \alpha(\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0.$$

Proof. See 3. **Q.E.D.**

3 Technical Details of Solution

In the main manuscript, we have provided analytically tractable results for the optimal strategies. In this subsection, the associated technical details are given. To derive the main results, the most difficult part is to solve the stochastic optimization problem (3). More specifically, the problem is inclined to involve unwanted unlimited downside utilities with high possibilities. This is because the term in (3) that involves the Poisson intensity δ is highly likely to be caught up in $-\infty$ when borrowing is allowed against human capital, i.e., as wealth $W(t)$ approaches $-\xi I(t)/\beta_1$. The simplest approach to remedy this problem is to introduce a time-varying lower bound of wealth:

$$W(t) > -L(t) > -\frac{k\xi I(t)}{\beta_1}, \ \text{ for all } t \geq 0, \tag{15}$$

where $L(t)$ is a given nonnegative time-varying function, but not completely arbitrary. With the help of the lower bound of wealth in (15), now the problem (3) is well defined.

The State Price Density. Solving the stochastic optimization problem (3) involves many steps. One of the most important steps is to characterize the unique state price density in an incomplete market. This is a daunting task, as there exist infinitely many possible candidates for the state price density in our economic setting, where not only the market risk, but also the income shock give rise to market incompleteness. In order to address the issue associated with market incompleteness, we rely on the fictitious completion of Karatzas *et al.* (1991). Specifically, we come up with a fictitious asset, $S^f(t)$,

and its price dynamics are given by

$$dS^f(t) = \mu^f S^f(t)dt + \sigma^f S^f(t)dM(t),$$

where μ^f and σ^f are to be determined with the minimal local martingale measure and $M(t)$ is a compensated martingale process related to a Poisson shock. Specifically, the dynamics of $M(t)$ follow

$$dM(t) = -\delta dt + dN(t),$$

where $N(t)$ represents the Poisson shock. Given the following income-shock-adjusted intensity

$$\hat{\delta} \equiv \delta - \frac{\mu^f - r}{\sigma^f},$$

the state price densities are then characterized by

$$\xi^{\hat{\delta}}(t) \equiv \exp\Big\{ \ln\Big(\frac{\hat{\delta}}{\delta}\Big)\mathbf{1}_{\{\tau \le t\}} - (\hat{\delta} - \delta)t\Big\}H(t),$$

where **1** is an indicator function that gives 1 if the Poisson shock occurs at time t and 0 otherwise, $H(t)$ is the standard state price density in complete markets under no arbitrage and its dynamics are given by

$$dH(t) = -H(t)\{rdt + \theta^\top dZ(t)\}, \;\; H(0) = 1.$$

Having characterized the state price densities in the presence of the income shock, we introduce one important lemma to convert the dynamic wealth constraint in (4) into the static wealth constraint as follows.

Lemma 3.1. *The dynamic wealth constraint in (4) can be converted into the following static wealth constraint:*

$$E\Big[\int_0^\infty e^{-\hat{\delta}t} H(t)\Big(c(t) - \xi I(t) + \hat{\delta} W(t)\Big)dt\Big] \le w. \tag{16}$$

Proof. By applying Itô's formula to $d\big(e^{-rt}W(t)\big)$ yields

$$d\big(e^{-rt}W(t)\big) = -e^{-rt}\{c(t) - \xi I(t)\}dt + e^{-rt}\pi(t)^\top d\tilde{Z}(t), \tag{17}$$

where $\tilde{Z}$ is the Brownian motion process under the new martingale measure with respect to the state

price density $\xi^{\hat{\delta}}(t)$. By Girsanov's theorem, the new probability measure is defined by

$$\tilde{P}(A) \equiv \int_A e^{rt}\xi^{\hat{\delta}}(t,\omega)dP(\omega) \text{ for all } A \in \mathcal{F} \tag{18}$$

and the Brownian motion process $\tilde{Z}$ follows

$$\tilde{Z}(t) \equiv \theta dt + Z(t).$$

Integrating both sides of (17) from 0 to τ,

$$\int_0^\tau e^{-rt}\Big(c(t) - \xi I(t)\Big)dt + e^{-r\tau}W(\tau) = w + \int_0^\tau e^{-rt}\pi(t)^\top d\tilde{Z}(t).$$

Taking expectation $\tilde{E}$ under the new martingale measure,

$$\tilde{E}\Big[\int_0^\tau e^{-rt}\Big(c(t) - \xi I(t)\Big)dt + e^{-r\tau}W(\tau)\Big] \leq w.$$

Changing the martingale measure into the physical measure using the relationship of (18),

$$E\Big[\int_0^\tau \xi^{\hat{\delta}}(t)\Big(c(t) - \xi I(t)\Big)dt + \xi^{\hat{\delta}}(\tau)W(\tau)\Big] \leq w.$$

Integrating out the Poisson intensity δ with respect to τ using the conditional expectation completes the proof of the lemma. **Q.E.D.**

The Static Optimization Problem. With the help of Lemma 3.1, we solve the stochastic optimization problem (3) by using the martingale representation approach in an incomplete market (Karatzas *et al.*, 1991), thereby converting the problem into the following static optimization problem:

$$\sup_{(c,W)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K\frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \tag{19}$$

subject to

$$E\Big[\int_0^\infty e^{-\hat{\delta}t}H(t)\Big(c(t) - \xi I(t) + \hat{\delta}W(t)\Big)dt\Big] \leq w.$$

The optimization problem given by (19) seems to be almost impossible to be solved analytically. We introduce a lemma to reformulate the optimization problem as follows.

Lemma 3.2. *The static optimization problem of (19) is reformulated as*

$$\begin{aligned}
&\inf_{(\lambda,\hat{\delta})}\{J^{\hat{\delta}}(\lambda,I)+\lambda w\}\\
&=\inf_{\lambda}\{\inf_{\hat{\delta}} J^{\hat{\delta}}(\lambda,I)+\lambda w\}\\
&\equiv\inf_{\lambda}\{J(\lambda,I)+\lambda w\},
\end{aligned} \tag{20}$$

where the indirect value function $J^{\hat{\delta}}(\lambda,I)$ is given by

$$\begin{aligned}
J^{\hat{\delta}}(\lambda,I)&=(\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^\infty e^{-(\beta_2+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(1+(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big)\Gamma^{\hat{\delta}}(t)^{1-1/\gamma}+\Big(1+\frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big]\\
&\equiv(\xi I)^{1-\gamma}\varphi_{\hat{\delta}}(z)
\end{aligned} \tag{21}$$

with $z=\lambda(\xi I)^\gamma$, where $\tilde{E}$ is the expectation under the new probability measure defined by

$$\tilde{P}(A)\equiv\int_A \exp\Big(-\frac{1}{2}(1-\gamma)^2||\sigma^I||^2(t,\omega)+(1-\gamma)(\sigma^I)^\top dZ(t,\omega)\Big)dP(\omega)\ \textit{for all}\ \ A\in\mathcal{F}$$

with the new Brownian motion process $\tilde{Z}$ given by

$$\tilde{Z}(t)=-(1-\gamma)\sigma^I dt+Z(t),$$

$\Gamma^{\hat{\delta}}(t)$ *is a new state variable defined by*

$$\Gamma^{\hat{\delta}}(t)\equiv\lambda e^{(\beta+\delta-\hat{\delta})t}H(t)(\xi I(t))^\gamma,$$

and its dynamics are given by

$$d\Gamma^{\hat{\delta}}(t)=\Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}}-\beta_2)dt+\beta_3 d\tilde{Z}(t)\},$$

$$\begin{aligned}
\beta_1^{\hat{\delta}}&=r-\mu^I+(\sigma^I)^\top\theta+\hat{\delta}-\delta,\\
\beta_2&=\beta-\mu^I(1-\gamma)+\frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,\\
\beta_3&=\gamma(\sigma^I)^\top-\theta^\top.
\end{aligned}$$

Proof. Using the standard Lagrangian approach, we can construct the indirect value function, $J^{\hat{\delta}}(\lambda,I)$,

and it is given by

$$J^{\hat{\delta}}(\lambda, I) \equiv \sup_{(c,W)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K \frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big] \\ -\lambda E\Big[\int_0^\infty e^{-\hat{\delta}t} H(t)\Big(c(t) - \xi I(t) + \hat{\delta} W(t)\Big)dt\Big]. \tag{22}$$

Applying the first-order conditions for consumption $c(t)$ and wealth $W(t)$ gives rise to

$$\begin{aligned} c(t) &= \Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\Big)^{-1/\gamma}, \\ W(t) &= \Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\Big)^{-1/\gamma}\Big(\frac{\hat{\delta}}{\delta}\Big)^{-1/\gamma} K^{1/\gamma} - k\xi I(t)/\beta_1. \end{aligned} \tag{23}$$

The indirect value function in (22) can be rewritten when the above first-order conditions for consumption and wealth are substituted in:

$$\begin{aligned} J^{\hat{\delta}}(\lambda, I) = E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big\{&\frac{\gamma}{1-\gamma}\Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\Big)^{1-1/\gamma} \\ &+ \frac{\gamma}{1-\gamma}(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\Big)^{1-1/\gamma} \\ &+ \Big(1 + \frac{\hat{\delta}k}{\beta_1}\Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\Big)\xi I(t)\Big)\Big\}dt\Big]. \end{aligned} \tag{24}$$

We introduce a new state variable to reformulate the indirect value function in (24). Specifically,

$$\Gamma^{\hat{\delta}}(t) \equiv \lambda e^{(\beta+\delta-\hat{\delta})t} H(t)(\xi I(t))^\gamma.$$

The indirect value function in (22) can be reformulated as the function of $\Gamma^{\hat{\delta}}(t)$:

$$J^{\hat{\delta}}(\lambda, I) = E\Big[\int_0^\infty (\xi I(t))^{1-\gamma} e^{-(\beta+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(\Gamma^{\hat{\delta}}(t)^{1-1/\gamma} + (\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Gamma^{\hat{\delta}}(t)^{1-1/\gamma}\Big) \\ + \Big(1 + \frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big].$$

By Girsanov's Theorem, the new probability measure can be defined by

$$\tilde{P}(A) \equiv \int_A \exp\Big(-\frac{1}{2}(1-\gamma)^2||\sigma^I||^2(t,\omega) + (1-\gamma)(\sigma^I)^\top dZ(t,\omega)\Big)dP(\omega) \text{ for all } A \in \mathcal{F}$$

and the new Brownian motion process $\tilde{Z}$ is given by

$$\tilde{Z}(t) = -(1-\gamma)\sigma^I dt + Z(t).$$

The dynamics of the new state variable follow

$$d\Gamma^{\hat{\delta}}(t) = \Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3 d\tilde{Z}(t)\},$$

where

$$\begin{aligned}
\beta_1^{\hat{\delta}} &= r - \mu^I + (\sigma^I)^\top \theta + \hat{\delta} - \delta, \\
\beta_2 &= \beta - \mu^I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2, \\
\beta_3 &= \gamma(\sigma^I)^\top - \theta^\top.
\end{aligned}$$

As a result, the indirect value function is given by

$$\begin{aligned}
J^{\hat{\delta}}(\lambda, I) &= (\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^\infty e^{-(\beta_2+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(1 + (\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big)\Gamma^{\hat{\delta}}(t)^{1-1/\gamma} + \Big(1 + \frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big] \\
&\equiv (\xi I)^{1-\gamma}\varphi_{\hat{\delta}}(z),
\end{aligned}$$

where $z = \lambda(\xi I)^\gamma$. Following Karatzas *et al.* (1999), the stochastic optimization problem (3) or equivalently, the static optimization problem (19) essentially derives from the indirect value function in (22) by the following relationship:

$$\begin{aligned}
V(w, I) &= \inf_{(\lambda, \hat{\delta})}\{J^{\hat{\delta}}(\lambda, I) + \lambda w\} \\
&= \inf_{\lambda}\{\inf_{\hat{\delta}} J^{\hat{\delta}}(\lambda, I) + \lambda w\} \\
&\equiv \inf_{\lambda}\{J(\lambda, I) + \lambda w\},
\end{aligned}$$

which completes the proof. **Q.E.D.**

The Income-Shock-Adjusted Intensity. In the lemma, $\hat{\delta}$ is to be determined to find out the minimal local martingale measure, guaranteeing the uniqueness of the state price density through which the risk neutral probability measure can be constructed to give more weight to unwanted events resulting from an unexpected, large, and negative income shock relative to the physical (or the original) probability

measure. The following lemma determines the unique $\hat{\delta}$.

Lemma 3.3. *The income-shock-adjusted intensity $\hat{\delta}$ is determined uniquely by*

$$\hat{\delta} = \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{z}, \tag{25}$$

where $G(z) = -\varphi'_\delta(z) + 1/\beta_1$ solves to the following non-linear differential equation:

$$\begin{aligned} -\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) \\ + \beta_1 G(z) + \delta K \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \ \ 0 < z < \overline{z}. \end{aligned} \tag{26}$$

with the boundary conditions

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \ \ \textit{and} \ \ G'(\overline{z}) = 0.$$

Proof. The function $\varphi_{\hat{\delta}}(z)$ in (21) should satisfy the following non-linear ordinary differential equation by Feynman-Kac's formula:

$$\begin{aligned} \inf_{\hat{\delta}} \Big[\frac{1}{2}||\beta_3||^2 z^2 \varphi''_{\hat{\delta}}(z) - (\beta_1^{\hat{\delta}} - \beta_2) z \varphi'_{\hat{\delta}}(z) - (\beta_2 + \delta)\varphi_{\hat{\delta}}(z) \\ + \frac{\gamma}{1-\gamma}\Big(1 + (\delta K)^{1/\gamma} \hat{\delta}^{1-1/\gamma}\Big) z^{1-1/\gamma} + \Big(1 + \frac{\hat{\delta} k}{\beta_1}\Big) z \Big] = 0, \ \ 0 < z < \overline{z}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \beta_1^{\hat{\delta}} &= r - \mu^I + (\sigma^I)^\top \theta + \hat{\delta} - \delta, \\ \beta_2 &= \beta - \mu^I (1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2, \\ \beta_3 &= \gamma(\sigma^I)^\top - \theta^\top, \end{aligned}$$

and $\overline{z}$ is to be determined according to the boundary conditions (or the value matching and smooth pasting conditions) given by

$$\varphi'_{\hat{\delta}}(\overline{z}) = \frac{L}{\xi I}, \ \ \varphi''_{\hat{\delta}}(\overline{z}) = 0.$$

Note that the technical details behind the boundary conditions stated above are essentially the same as Dybvig and Liu (2011). Applying the first-order condition for $\hat{\delta}$ leads to

$$\hat{\delta} = \Big(-\varphi'_{\hat{\delta}}(z) + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{z}. \tag{28}$$

When the above first-order condition is substituted in (27), the differential equation is rewritten as

$$\frac{1}{2}||\beta_3||^2 z^2 \varphi''_{\hat{\delta}}(z) - (\beta_1 - \delta - \beta_2) z \varphi'_{\hat{\delta}}(z) - (\beta_2 + \delta)\varphi_{\hat{\delta}}(z) \\ + \frac{\gamma}{1-\gamma} z^{1-1/\gamma} + z + \frac{\delta K}{1-\gamma}\Big(- \varphi'_{\hat{\delta}}(z) + \frac{k}{\beta_1}\Big)^{1-\gamma} = 0, \;\; 0 < z < \overline{z}. \tag{29}$$

From now on, we will carry out several transformations to simplify the differential equation of (29). We denote $-\varphi'_{\hat{\delta}}(z)$ by $\tilde{G}(z)$. By differentiating the both sides of (29) with respect to z, the differential equation (29) is restated with $\tilde{G}(z)$ as follows:

$$-\frac{1}{2}||\beta_3||^2 z^2 \tilde{G}''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z \tilde{G}'(z) \\ + \beta_1 \tilde{G}(z) + 1 + \delta K\Big(\tilde{G}(z) + \frac{k}{\beta_1}\Big)^{-\gamma} \tilde{G}'(z) = z^{-1/\gamma}, \;\; 0 < z < \overline{z}, \tag{30}$$

with the boundary conditions

$$\tilde{G}(\overline{z}) = -\frac{L}{\xi I} \;\text{ and }\; \tilde{G}'(\overline{z}) = 0.$$

We also denote $\tilde{G}(z) + 1/\beta_1$ by $G(z)$. Then the differential equation (30) is rewritten as

$$-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) \\ + \beta_1 G(z) + \delta K\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \;\; 0 < z < \overline{z}.$$

with the boundary conditions

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \;\text{ and }\; G'(\overline{z}) = 0.$$

Finally, $\hat{\delta}$ given in (28) is rewritten as a function of $G(z)$, which completes the proof. **Q.E.D.**

The Indirect Value Function. The relationship (20) shows that the optimization problem in (3) or equivalently, the problem (19) is solved by deriving the indirect value function in (21) together with the income-shock-adjusted intensity $\hat{\delta}$ in the lemma. Going forward, we devote our full attention to solving the non-linear differential equation of (26) to derive the indirect value function $J(\lambda, I)$ in (21).

Now, we introduce the important proposition to derive a general solution to the differential equation of (26).

Proposition 3.1. *A general solution to the differential equation of (26) is given by*

$$\begin{aligned} G(z) =& \frac{1}{\hat{A}+\delta} z^{-1/\gamma} + B^*(\overline{z};\delta) z^{-\alpha_\delta^*} \\ &+ \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} \Big[(\alpha_\delta - 1) z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2} \Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} d\mu \\ &+ (\alpha_\delta^* - 1) z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2} \Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} d\mu \Big], \end{aligned} \tag{31}$$

where $\hat{A}$ is given by

$$\hat{A} = \frac{\gamma - 1}{\gamma} \Big(\beta_1 + \frac{||\beta_3||^2}{2\gamma} \Big) + \frac{\beta_2}{\gamma},$$

$B^*(\overline{z};\delta)$ and $\overline{z}$ are the two constants to be determined by the boundary conditions:

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \ \text{ and } \ G'(z) = 0,$$

and $\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots to the following characteristic equations:

$$F(\alpha;\delta) \equiv -\frac{1}{2}||\beta_3||^2 \alpha(\alpha - 1) + \alpha(\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0.$$

Proof. We conjecture a general solution of the equation (26) as

$$G(z) = \frac{1}{\hat{A}+\delta} z^{-1/\gamma} + \eta(z) z^{-\alpha_\delta} + \eta^*(z) z^{-\alpha_\delta^*}, \tag{32}$$

subject to

$$\eta'(z) z^{-\alpha_\delta} + (\eta^*(z))' z^{-\alpha_\delta^*} = 0,$$

where $\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots to the following characteristic equations:

$$F(\alpha;\delta) \equiv -\frac{1}{2}||\beta_3||^2 \alpha(\alpha - 1) + \alpha(\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0.$$

Direct calculations of the first and second derivative of G result in

$$G'(z) = -\frac{1}{\gamma(\hat{A}+\delta)} z^{-1/\gamma - 1} - \alpha_\delta \eta(z) z^{-\alpha_\delta - 1} - \alpha_\delta^* \eta^*(z) z^{-\alpha_\delta^* - 1}$$

and

$$G''(z) = \Big(1+\frac{1}{\gamma}\Big)\frac{1}{\gamma(\hat{A}+\delta)}z^{-1/\gamma-2} - \alpha_\delta \eta'(z) z^{-\alpha_\delta-1} + \alpha_\delta(\alpha_\delta+1)\eta(z)z^{-\alpha_\delta-2}$$
$$- \alpha_\delta^*(\eta^*(z))' z^{-\alpha_\delta^*-1} + \alpha_\delta^*(\alpha_\delta^*+1)\eta^*(z)z^{-\alpha_\delta^*-2}.$$

Using the general solution (32) and the derivatives of G stated above, the first three terms of left-hand side in (26) become

$$\begin{aligned}
&-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2+\beta_2+\delta-\beta_1)zG'(z) + \beta_1 G(z)\\
&= z^{-1/\gamma} + \frac{||\beta_3||^2}{2}(\alpha_\delta-\alpha_\delta^*)z^{1-\alpha_\delta}\eta'(z)\\
&= z^{-1/\gamma} - \frac{||\beta_3||^2}{2}(\alpha_\delta-\alpha_\delta^*)z^{1-\alpha_\delta^*}(\eta^*(z)).
\end{aligned}$$

As a result, the differential equation (26) simplifies to the following: for $0 < z < \overline{z}$,

$$\frac{||\beta_3||^2}{2}(\alpha_\delta-\alpha_\delta^*)z^{1-\alpha_\delta}\eta'(z) = -\delta K\Big(G(z)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(z)$$

and

$$\frac{||\beta_3||^2}{2}(\alpha_\delta-\alpha_\delta^*)z^{1-\alpha_\delta^*}(\eta^*(z)) = \delta K\Big(G(z)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(z).$$

Integrating the both sides of the above two relationships from 0 to z and from z to $\overline{z}$ allows $\eta(z)$ and $\eta^*(z)$ to be expressed as an integral form:

$$\eta(z) = -\frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)}\int_0^z \mu^{\alpha_\delta-1}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)d\mu$$

and

$$\eta^*(z) = \eta^*(\overline{z}) - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-1}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)d\mu.$$

Therefore, the general solution (32) also can be expressed as an integral form:

$$\begin{aligned}
G(z) = \frac{1}{\hat{A}+\delta}z^{-1/\gamma} + \eta^*(\overline{z})z^{-\alpha_\delta^*} - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)}\Big[z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta-1}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)d\mu\\
+ z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-1}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)d\mu\Big].
\end{aligned} \tag{33}$$

Note that

$$\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)=\frac{d}{d\mu}\Big\{\frac{1}{1-\gamma}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}\Big\}.$$

Using the integration by parts, the general solution (33) can be restated as follows:

$$\begin{aligned}G(z)=&\frac{1}{\hat{A}+\delta}z^{-1/\gamma}+\Big\{\eta^*(\overline{z})+\overline{z}^{\alpha_\delta^*-1}\frac{1}{1-\gamma}\Big(G(\overline{z})-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}\Big\}z^{-\alpha_\delta^*}\\&+\frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)z^{-\alpha_\delta}\int_0^z\mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\\&+(\alpha_\delta^*-1)z^{-\alpha_\delta^*}\int_z^{\overline{z}}\mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\Big].\end{aligned}\tag{34}$$

Defining a constant $B^*(\overline{z};\delta)$ as

$$B^*(\overline{z};\delta)\equiv\eta^*(\overline{z})+\overline{z}^{\alpha_\delta^*-1}\frac{1}{1-\gamma}\Big(G(\overline{z})-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}.$$

Finally, we obtain the general solution in closed-form:

$$\begin{aligned}G(z)=&\frac{1}{\hat{A}+\delta}z^{-1/\gamma}+B^*(\overline{z};\delta)z^{-\alpha_\delta^*}\\&+\frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)z^{-\alpha_\delta}\int_0^z\mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\\&+(\alpha_\delta^*-1)z^{-\alpha_\delta^*}\int_z^{\overline{z}}\mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\Big],\end{aligned}$$

which completes the proof. **Q.E.D.**

Optimal Consumption and Investment Strategies. Now, we are ready to derive the analytic results of optimal consumption and investment strategies.

Theorem 3.1. *The optimal strategies for consumption $c(t)$ and risky stock investment $\pi(t)$ are obtained in closed-form as follows:*

$$\begin{aligned}c(t)=(\hat{A}+\delta)\Big[&w+\frac{\xi I}{\beta_1}-\xi IB^*(\overline{z};\delta)z^{-\alpha_\delta^*}\\&-\textit{\{income risk-driven precautionary savings\}}\Big],\end{aligned}\tag{35}$$

$$
\begin{aligned}
\pi(t) &= \sigma^{-1}(\beta_3)^\top \xi I G'(z) z + \sigma^{-1}\sigma^I w \\
&= \sigma^{-1}(\gamma\sigma^I - \theta)\xi I G'(z) z + \sigma^{-1}\sigma^I w \\
&= \frac{1}{\gamma}\sigma^{-1}\theta w \\
&\quad + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\Big[\frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi I B^*(\overline{z};\delta) z^{-\alpha_\delta^*} \\
&\quad - \frac{2\gamma}{||\beta_3||^2}\delta K \frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/ c(t)^{-\gamma} \\
&\quad + (\gamma\alpha_\delta - 1) \times \textit{\{first term of income risk-driven precautionary savings\}} \\
&\quad + (\gamma\alpha_\delta^* - 1) \times \textit{\{second term of income risk-driven precautionary savings\}}\Big],
\end{aligned} \tag{36}
$$

where the income risk-driven precautionary savings are given by

$$
\begin{aligned}
\frac{2\delta K \xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big[&(\alpha_\delta - 1) z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\
&+ (\alpha_\delta^* - 1) z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu\Big].
\end{aligned}
$$

For simplicity, the income risk-driven precautionary savings are rewritten as

$$
\begin{aligned}
&\textit{\{income risk-driven precautionary savings\}} \\
&= \textit{\{first term of income risk-driven precautionary savings\}} \\
&\quad + \textit{\{second term of income risk-driven precautionary savings\}},
\end{aligned} \tag{37}
$$

where the first term is given by

$$
\frac{2\delta K(\alpha_\delta - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu < 0,
$$

and the second term is given by

$$
\frac{2\delta K(\alpha_\delta^* - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu > 0.
$$

Proof. With the first-order condition (25) for $\hat{\delta}$, the first-order conditions for consumption $c(t)$ in (23) can be rewritten as

$$
c(t) = \xi I(t)\Gamma^{\hat{\delta}}(t)^{-1/\gamma}, \tag{38}
$$

where $\Gamma^{\hat{\delta}}(t)$ is given by

$$\Gamma^{\hat{\delta}(t)} = \lambda e^{(\beta+\delta-\hat{\delta})t} H(t)(\xi I(t))^{\gamma},$$

and

$$\hat{\delta} = \Big(G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{\Gamma^{\hat{\delta}}(t)}$$

By the principle of dynamic programming, it is convenient to express the consumption as a function of initial variable z:

$$c(t) = c(0) = \xi I z^{-1/\gamma}. \tag{39}$$

From the relationship (20) between the value function and the indirect value function, applying the first-order condition for λ results in

$$\begin{aligned} w &= -J_{\lambda}(\lambda, I) \\ &= -\xi I \varphi'_{\hat{\delta}}(z) \\ &= \xi I \tilde{G}(z) \\ &= \xi I \Big(G(z) - \frac{1}{\beta_1}\Big), \end{aligned} \tag{40}$$

accordingly,

$$G(z) = \frac{w}{\xi I} + \frac{1}{\beta_1}.$$

A little rearrangement of the general solution (31) leads to

$$\begin{aligned} z^{-1/\gamma} = (\hat{A}+\delta)\Big[&G(z) - B^*(\overline{z};\delta) z^{-\alpha^*_\delta} \\ &- \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha^*_\delta)(1-\gamma)}\Big[(\alpha_\delta - 1) z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\ &+ (\alpha^*_\delta - 1) z^{-\alpha^*_\delta} \int_z^{\overline{z}} \mu^{\alpha^*_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu\Big] \\ = (\hat{A}+\delta)\Big[&\frac{w}{\xi I} + \frac{1}{\beta_1} - B^*(\overline{z};\delta) z^{-\alpha^*_\delta} \\ &- \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha^*_\delta)(1-\gamma)}\Big[(\alpha_\delta - 1) z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\ &+ (\alpha^*_\delta - 1) z^{-\alpha^*_\delta} \int_z^{\overline{z}} \mu^{\alpha^*_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu\Big]. \end{aligned}$$

Therefore, the first-order condition for consumption $c(t)$ in (38) allows the following optimal con-

sumption strategy:

$$c(t) = (\hat{A} + \delta)\Big[w + \frac{\xi I}{\beta_1} - \xi I B^*(\overline{z};\delta) z^{-\alpha_\delta^*} \\ - \{\text{income risk-driven precautionary savings}\}\Big],$$

where the income risk-driven precautionary savings are given by

$$\frac{2\delta K \xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta - 1) z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\ + (\alpha_\delta^* - 1) z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu\Big].$$

For simplicity, the income risk-driven precautionary savings are rewritten as

$$\begin{aligned} &\{\text{income risk-driven precautionary savings}\} \\ &= \{\text{first term of income risk-driven precautionary savings}\} \\ &\quad + \{\text{second term of income risk-driven precautionary savings}\}, \end{aligned}$$

where the first term is given by

$$\frac{2\delta K(\alpha_\delta - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu < 0,$$

and the second term is given by

$$\frac{2\delta K(\alpha_\delta^* - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu > 0.$$

It remains to derive the optimal investment strategy. A little rearrangement of the relationship in (40) gives

$$\frac{w}{\xi I} = G(z) - \frac{1}{\beta_1},$$

or equivalently,

$$\frac{W(t)}{\xi I(t)} = G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1}. \tag{41}$$

By applying It$\hat{o}$'s formula to the left hand side of the above relationship,

$$
\begin{aligned}
d\Big(\frac{W(t)}{\xi I(t)}\Big) &= dW(t)\frac{1}{\xi I(t)} + W(t)d\Big(\frac{1}{\xi I(t)}\Big) + dW(t)d\Big(\frac{1}{\xi I(t)}\Big) \\
&= \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big]\frac{1}{\xi I(t)} \\
&\quad + W(t)\Big[-(\xi I(t))^{-2}dI(t) + (\xi I(t))^{-3}(dI(t))^2\Big] \\
&\quad + \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big] \\
&\quad \times \Big[-(\xi I(t))^{-2}dI(t) + (\xi I(t))^{-3}(dI(t))^2\Big] \\
&= \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big]\frac{1}{\xi I(t)} \\
&\quad + W(t)\Big[-(\xi I(t))^{-1}\{\mu^I dt + (\sigma^I)^\top dZ(t)\} + (\xi I(t))^{-1}||\sigma^I||^2 dt\Big] \\
&\quad + \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big] \\
&\quad \times \Big[-(\xi I(t))^{-1}\{\mu^I dt + (\sigma^I)^\top dZ(t)\} + (\xi I(t))^{-1}||\sigma^I||^2 dt\Big] \\
&= \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big]\frac{1}{\xi I(t)} \\
&\quad + \frac{W(t)}{\xi I(t)}\Big[-(\mu^I - ||\sigma^I||^2)dt - (\sigma^I)^\top dZ(t)\Big] - (\xi I(t))^{-1}\pi(t)^\top\sigma^\top\sigma^I dt \\
&= \frac{1}{\xi I(t)}\Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1}) - W(t)(\mu^I - ||\sigma^I||^2) - \pi(t)^\top\sigma^\top\sigma^I\Big\}dt \\
&\qquad + \{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top\}dZ(t)\Big] \\
&= \frac{1}{\xi I(t)}\Big[\Big\{\{r - \mu^I + ||\sigma^I||^2\}W(t) - c(t) + \xi I(t) + \pi(t)^\top\{\mu - r\mathbf{1} - \sigma^\top\sigma^I\}\Big\}dt \\
&\qquad + \{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top\}dZ(t)\Big].
\end{aligned}
\tag{42}
$$

By applying It$\hat{o}$'s formula to the right hand side of the relationship (41),

$$
\begin{aligned}
dG(\Gamma^{\hat{\delta}}(t)) &= G'(\Gamma^{\hat{\delta}}(t))d\Gamma^{\hat{\delta}}(t) + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(d\Gamma^{\hat{\delta}}(t))^2 \\
&= G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3 d\tilde{Z}(t)\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2 dt \\
&= G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3\{-(1-\gamma)\sigma^I dt + dZ(t)\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2 dt \\
&= \Big\{-G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{(\beta_1^{\hat{\delta}} - \beta_2) + (1-\gamma)\beta_3\sigma^I\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2\Big\}dt \\
&\quad + G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\beta_3 dZ(t).
\end{aligned}
\tag{43}
$$

Equating each term of $dZ(t)$ in $d(W(t)/(\xi I(t)))$ and $dG(\Gamma^{\hat{\delta}}(t))$ derives the following relationship

that involves the optimal investment strategy $\pi(t)$:

$$\frac{\pi(t)^\top \sigma^\top - W(t)(\sigma^I)^\top}{\xi I(t)} = G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\beta_3. \tag{44}$$

By the principle of dynamic programming, it is convenient to express the investment as a function of initial variables at time 0:

$$\frac{\pi^\top \sigma^\top - w(\sigma^I)^\top}{\xi I} = G'(z)z\beta_3, \tag{45}$$

where $\pi = \pi(t) = \pi(0)$. Using the general solution $G(z)$ given in (31), a direct calculation of $G'(z)$ yields

$$\begin{aligned} G'(z) = &- \frac{1}{\gamma(\hat{A}+\delta)} z^{-1/\gamma-1} - \alpha_\delta^* B^*(\overline{z};\delta) z^{-\alpha_\delta^*-1} + \frac{2\delta K}{||\beta_3||^2(1-\gamma)z^2}\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} \\ &- \frac{2\delta K \alpha_\delta(\alpha_\delta - 1)}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta-1} \int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\ &- \frac{2\delta K \alpha_\delta^*(\alpha_\delta^* - 1)}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta^*-1} \int_0^z \mu^{\alpha_\delta^*-2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu. \end{aligned}$$

Multiply $G'(z)$ by $\xi I z$ gives

$$\begin{aligned} \xi I G'(z)z = &- \frac{1}{\gamma(\hat{A}+\delta)} \xi I z^{-1/\gamma} - \alpha_\delta^* \xi I B^*(\overline{z};\delta) z^{-\alpha_\delta^*} + \frac{2\delta K \xi I}{||\beta_3||^2(1-\gamma)z}\Big(G(z) - \frac{1}{\beta_1 + \frac{k}{\beta_1}} + \Big)^{1-\gamma} \\ &- \alpha_\delta \times \{\text{first term of income risk-driven precautionary savings}\} \\ &- \alpha_\delta^* \times \{\text{second term of income risk-driven precautionary savings}\}, \end{aligned}$$

where the first and second term of income risk-driven precautionary savings are given in (37). Note that $\xi I z^{-1/\gamma}$ of the first term in the above relationship is equivalent to the optimal consumption strategy from (39), as a result, $\xi I G'(z)z$ can be restated with (35) as the following:

$$\begin{aligned} \xi I G'(z)z = -\frac{1}{\gamma}\Big[& w + \frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi I B^*(\overline{z};\delta) z^{-\alpha_\delta^*} \\ & - \frac{2\gamma}{||\beta_3||^2(1-\gamma)z} \xi I \delta K \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} \\ & + (\gamma\alpha_\delta - 1) \times \{\text{first term of income risk-driven precautionary savings}\} \\ & + (\gamma\alpha_\delta^* - 1) \times \{\text{second term of income risk-driven precautionary savings}\}\Big]. \end{aligned}$$

Note that

$$
\begin{aligned}
&-\frac{2\gamma}{||\beta_3||^2(1-\gamma)z}\xi I\delta K\Big(G(z)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}\\
&=-\frac{2\gamma}{||\beta_3||^2 z}\xi I\delta K\frac{\Big(\frac{w}{\xi I}+\frac{k}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\\
&=-\frac{2\gamma}{||\beta_3||^2 z}(\xi I)^{\gamma}\delta K\frac{\Big(w+\frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\\
&=-\frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w+\frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}c(t)^{\gamma}\\
&=-\frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w+\frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/c(t)^{-\gamma}.
\end{aligned}
$$

Therefore, we derive the optimal investment strategy from (45):

$$
\begin{aligned}
\pi(t)&=\sigma^{-1}(\beta_3)^{\top}\xi IG'(z)z+\sigma^{-1}\sigma^{I}w\\
&=\sigma^{-1}(\gamma\sigma^{I}-\theta)\xi IG'(z)z+\sigma^{-1}\sigma^{I}w\\
&=\frac{1}{\gamma}\sigma^{-1}\theta w\\
&\quad+\frac{1}{\gamma}\sigma^{-1}(\theta-\gamma\sigma^{I})\Big[\frac{\xi I}{\beta_1}+(\gamma\alpha_{\delta}^{*}-1)\xi IB^{*}(\overline{z};\delta)z^{-\alpha_{\delta}^{*}}\\
&\quad-\frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w+\frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/c(t)^{-\gamma}\\
&\quad+(\gamma\alpha_{\delta}-1)\times\{\text{first term of income risk-driven precautionary savings}\}\\
&\quad+(\gamma\alpha_{\delta}^{*}-1)\times\{\text{second term of income risk-driven precautionary savings}\}\Big].
\end{aligned}
$$

Until now, we have derived the optimal consumption and investment strategies, $c(t)$ and $\pi(t)$, in closed-form in (35) and (36), respectively, together with the income-shock-adjusted intensity $\hat{\delta}$ in (25). Following Karatzas *et al.* (1991), the optimality would be verified if the wealth process $W(t)$

was self financed by $c(t)$ and $\pi(t)$. The term of dt of $dG(\Gamma^{\hat{\delta}}(t))$ in (43) is rewritten as

$$
\begin{aligned}
&- G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{(\beta_1^{\hat{\delta}} - \beta_2) + (1-\gamma)\beta_3\sigma^I\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2 \\
&= \frac{1}{2}||\beta_3||^2(\Gamma^{\hat{\delta}}(t))^2G''(\Gamma^{\hat{\delta}}(t)) + \{\beta_2 - \beta_1 - \hat{\delta} + \delta - (1-\gamma)\beta_3\sigma^I\}\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) \\
&= \frac{1}{2}||\beta_3||^2(\Gamma^{\hat{\delta}}(t))^2G''(\Gamma^{\hat{\delta}}(t)) + (\beta_2 + \delta - \beta_1)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) - (1-\gamma)\beta_3\sigma^I\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) \\
&\quad - \Big(G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma}\frac{\delta K}{\Gamma^{\hat{\delta}}(t)}\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) \\
&= \frac{1}{2}||\beta_3||^2(\Gamma^{\hat{\delta}}(t))^2G''(\Gamma^{\hat{\delta}}(t)) + (\beta_2 + \delta - \beta_1)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) - \delta K\Big(G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma}G'(\Gamma^{\hat{\delta}}(t)) \\
&\quad - (1-\gamma)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))\beta_3\sigma^I \\
&= -||\beta_3||^2\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) + \beta_1 G(\Gamma^{\hat{\delta}}(t)) - \Gamma^{\hat{\delta}}(t)^{-1/\gamma} - (1-\gamma)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))\beta_3\sigma^I \\
&= -\frac{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top}{\xi I(t)}(\beta_3)^\top + \beta_1\Big(\frac{W(t)}{\xi I(t)} + \frac{1}{\beta_1}\Big) - \frac{c(t)}{\xi I(t)} - (1-\gamma)\frac{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top}{\xi I(t)}\sigma^I \\
&= \frac{1}{\xi I(t)}\Big[\{(\sigma^I)^\top(\beta_3)^\top + \beta_1\}W(t) - c(t) + \xi I(t) - \pi(t)^\top\sigma^\top(\beta_3)^\top - (1-\gamma)\pi(t)^\top\sigma^\top\sigma^I + (1-\gamma)||\sigma^I||^2W(t)\Big] \\
&= \frac{1}{\xi I(t)}\Big[\{(\sigma^I)^\top(\gamma\sigma^I - \theta) + r - \mu^I + (\sigma^I)^\top\theta\}W(t) - c(t) + \xi I(t) - \pi(t)^\top\sigma^\top(\gamma\sigma^I - \theta) \\
&\qquad - (1-\gamma)\pi(t)^\top\sigma^\top\sigma^I + (1-\gamma)||\sigma^I||^2W(t)\Big] \\
&= \frac{1}{\xi I(t)}\Big[\{r - \mu^I + \gamma||\sigma^I||^2\}W(t) - c(t) + \xi I(t) + \pi(t)^\top\{\mu - r\mathbf{1} - \sigma^\top\sigma^I\}\Big].
\end{aligned}
$$

where the second equality derives when $\hat{\delta}$ in (25) substituted in, the fourth equality derives from the differential equation in (26), the fifth equality derives from $||\beta_3||^2 = \beta(\beta_3)^\top$, (38), (41), and (44). This shows that each term of dt in $d(W(t)/(\xi I(t)))$ and $dG(\Gamma^{\hat{\delta}}(t))$ are exactly the same, as a result, the wealth process $W(t)$ is self financed by the optimal consumption strategy $c(t)$ and the optimal investment strategy $\pi(t)$ with the income-shock-adjusted intensity $\hat{\delta}$ in (25). **Q.E.D.**

4 Technical Details of a General Equilibrium Analysis

In the main manuscript, we have derived the general equilibrium quantities in the presence of the large, negative income shock. We consider a simple exchange economy in the style of Lucas (1978). The economy is populated by a representative agent facing an unexpected, large, and negative income shock. The agent is entitled to an endowment to be consumed in equilibrium and is assumed to trade a riskless bond and multiple risky stocks distributing the dividend. The returns to these assets adjust

to represent a no-trade equilibrium. The risk-free interest rate, r, the constant mean vector, μ, and the constant nonsingular standard deviation matrix, σ, should be determined from the equilibrium conditions, as specified below:

Definition 4.1. *An equilibrium can be characterized as a collection of* (r, μ, σ) *and optimal strategies* $(c(t), \pi(t))$ *such that the consumption good, stock, and bond markets clear as*

$$\begin{aligned} &c(t) = D(t), \\ &\pi^j(t) = S^j(t), \;\; j = 1, ..., N, \\ &W(t) = \sum_{j=1}^{N} S^j(t), \end{aligned}$$

where $D(t) = \sum_{j=1}^{N} d_j$ *is the aggregate dividend process and* N *is the number of risky stocks.*

The following proposition provides the unique state price density in the presence of the market risk and the income shock.

Proposition 4.1. *The unique state price density is given by*

$$\xi^{\hat{\delta}}(t) = \exp\Big\{ \ln\Big(\frac{\hat{\delta}}{\delta}\Big)\boldsymbol{1}_{\{\tau=t\}} - (\hat{\delta} - \delta)t\Big\} H(t), \tag{46}$$

where

$$\hat{\delta} = \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{z},$$

τ *is the arrival time of a Poisson shock,* $\boldsymbol{1}$ *is an indicator function that gives* 1 *if the Poisson shock occurs at time* t *and* 0 *otherwise,* $G(z)$ *satisfies the following differential equation:*

$$\begin{aligned} -\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) \\ + \beta_1 G(z) + \delta K\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \;\; 0 < z < \overline{z}, \end{aligned} \tag{47}$$

with

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1}, G'(\overline{z}) = 0,$$

and the dynamics of $H(t)$ *are given by*

$$dH(t) = -H(t)\{r dt + \theta^\top dZ(t)\}, \;\; H(0) = 1.$$

Proof. See 3. **Q.E.D.**

Given the unique state price density, the following proposition solves for the equilibrium state price density, and the equilibrium risk-free interest rate and the equilibrium Sharpe ratio.

Proposition 4.2. *The equilibrium state price density is given by*

$$H(t) = \frac{1}{\lambda} e^{-(\beta-(\hat{\delta}(r)-\delta))t} (\xi I(t))^{-\gamma}, \tag{48}$$

the equilibrium risk-free interest rate and the equilibrium Sharpe ratio are given by

$$r = \beta + \gamma\mu^I - \frac{1}{2}\gamma(1+\gamma)(\sigma^I)^2 - (\hat{\delta}(r) - \delta) \tag{49}$$

and

$$\theta = \gamma\sigma^I, \tag{50}$$

respectively, where μ^I and σ^I represent the expected consumption growth rate and volatility of consumption growth rate, and the constant $\hat{\delta}(r)$ is determined by solving the following non-linear algebraic equation:

$$\hat{\delta}(r) = \Big\{\Big(\frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))}\Big)\Big/\Big(\frac{w}{\xi I} + \frac{k}{\beta_1(\hat{\delta}(r))}\Big)\Big\}^{\gamma} \{\beta_1(\hat{\delta}(r))\}^{\gamma} \delta K(r)$$

with

$$\beta_1(\hat{\delta}(r)) = \beta + (\gamma-1)\mu^I - \frac{1}{2}\gamma(\gamma-1)(\sigma^I)^2 - (\hat{\delta}(r) - \delta),$$

$$K(r) = \Big\{\frac{\gamma-1}{\gamma}\Big(r + \frac{\gamma(\sigma^I)^2}{2}\Big) + \frac{\beta}{\gamma}\Big\}^{-\gamma},$$

and the constant λ satisfies

$$E\Big[\int_0^\infty e^{-\hat{\delta}(r)t} H(t)\Big\{c(t) - \xi I(t) + \hat{\delta}(r)W(t)\Big\}dt\Big] = w, \tag{51}$$

$$c(t) = \Big(\lambda e^{(\beta+\delta-\hat{\delta}(r))t} H(t)\Big)^{-1/\gamma},\ W(t) = \Big(\lambda e^{(\beta+\delta-\hat{\delta}(r))t} H(t)\Big)^{-1/\gamma}\Big(\frac{\hat{\delta}(r)}{\delta}\Big)^{-1/\gamma} K^{1/\gamma} - \frac{k\xi I(t)}{\beta_1(\hat{\delta}(r))},$$

with (48), (49), and (50) substituted in.

Proof. The optimal consumption strategy derives from (23):

$$c(t) = \Big(\lambda e^{(\beta+\delta-\hat{\delta}(r))t} H(t)\Big)^{-1/\gamma},$$

where the constant λ should satisfy

$$E\Big[\int_0^\infty e^{-\hat{\delta}(r)t} H(t)\Big\{c(t) - \xi I(t) + \hat{\delta}(r)W(t)\Big\}dt\Big] = w$$

with the following optimal wealth process:

$$W(t) = \Big(\lambda e^{(\beta+\delta-\hat{\delta}(r))t} H(t)\Big)^{-1/\gamma}\Big(\frac{\hat{\delta}(r)}{\delta}\Big)^{-1/\gamma} K^{1/\gamma} - \frac{k\xi I(t)}{\beta_1(\hat{\delta}(r))},$$

and $\hat{\delta}(r)$ is given by Proposition 4.1 as follows:

$$\hat{\delta}(r) = \Big(G(z) - \frac{1}{\beta_1(\hat{\delta}(r))} + \frac{k}{\beta_1(\hat{\delta}(r))}\Big)^{-\gamma}\frac{\delta K}{z}, \tag{52}$$

where $G(z)$ solves the differential equation of (47) with

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))}, G'(\overline{z}) = 0.$$

According to the clearing condition of consumption good, $c(t) = D(t)$, the equilibrium state price density $H(t)$ follows

$$H(t) = \frac{1}{\lambda} e^{-(\beta-(\hat{\delta}(r)-\delta))t} D(t)^{-\gamma},$$

or equivalently,

$$H(t) = \frac{1}{\lambda} e^{-(\beta-(\hat{\delta}(r)-\delta))t} (\xi I(t))^{-\gamma}. \tag{53}$$

Applying It$\hat{o}$'s formula to the both sides of (53),

$$\begin{aligned}
&- H(t)\{rdt + \theta^\top dZ(t)\} \\
&= -H(t)\Big\{\Big(\beta - (\hat{\delta}(r) - \delta) + \gamma\mu^I - \frac{1}{2}\gamma(1+\gamma)||\sigma^I||^2\Big)dt + \gamma(\sigma^I)^\top dZ(t)\Big\}.
\end{aligned}$$

Equating each term of dt and $dZ(t)$ gives the equilibrium risk-free interest rate and the equilibrium Sharpe ratio.

With the equilibrium quantities (49) and (50), and the clearing conditions of stock and bond markets, the differential equation of (47) has a solution in closed-form:

$$G(z) = \frac{1}{\beta_1(\hat{\delta}(r))} z^{-1/\gamma}.$$

By substituting the solution $G(z)$ in (52), $\hat{\delta}(r)$ is determined by

$$\hat{\delta}(r) = \Big(\frac{1}{\beta_1(\hat{\delta}(r))} z^{-1/\gamma} - \frac{1}{\beta_1(\hat{\delta}(r))} + \frac{k}{\beta_1(\hat{\delta}(r))}\Big)^{-\gamma} \frac{\delta K}{z}. \tag{54}$$

Recall the relationship (40)

$$w = \xi I\Big(G(z) - \frac{1}{\beta_1(\hat{\delta}(r))}\Big),$$

accordingly,

$$\frac{1}{\beta_1(\hat{\delta}(r))} z^{-1/\gamma} = \frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))},$$

or equivalently,

$$z^{-1} = \beta_1^{\gamma}\Big(\frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))}\Big)^{\gamma}.$$

As a result, the equation (54) reduces to

$$\hat{\delta}(r) = \Big\{\Big(\frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))}\Big)\Big/\Big(\frac{w}{\xi I} + \frac{k}{\beta_1(\hat{\delta}(r))}\Big)\Big\}^{\gamma} \beta_1^{\gamma} \delta K$$

with

$$\begin{aligned}
\beta_1(\hat{\delta}(r)) &= r - \mu^I + (\sigma^I)^\top \theta \\
&= \beta + \gamma\mu^I - \frac{1}{2}\gamma(1+\gamma)||\sigma^I||^2 - (\hat{\delta}(r) - \delta) - \mu^I + (\sigma^I)^\top \theta \\
&= \beta + (\gamma - 1)\mu^I - \frac{1}{2}\gamma(\gamma - 1)||\sigma^I||^2 - (\hat{\delta}(r) - \delta),
\end{aligned}$$

where the second equality derives from the substitution of (49) and the third equality comes from the substitution of (50). This completes the proof. **Q.E.D.**

5 Technical Details of Quantitative Analysis

Data Description. The data on wealth and income are from family net worth and before tax family income by selected characteristics of families in the SCF for the period of 1995-2010. We sort the family net worth and before tax family income into age groups and percentile of net worth, and use

them to compute the normalized cash-on-hand, which is the ratio of net worth to income. We find from the data that the family net worth (before tax family income) by age groups shows a hump-shaped profile; until age of 55-64 (45-54) the net worth (the family income) increases with age, but subsequently falls with age on average. The net worth and the family income increase with an increase in percentile of net worth.

Lower Bound of Wealth. Recall that the lower bound $-L(t)$ with $L(0) = L$ of wealth in (15) can be empirically plausible by using the following relationship:

$$L = \omega \frac{r + \nu + \delta k}{(r + \nu + \delta)(r + \nu)} \epsilon, \text{ for } 0 \leq \omega < 1, \tag{55}$$

where $\epsilon = \xi I$ represents a constant stream of earnings over the life cycle, ω denotes the extent to which credit is tightened, and ν is an individual's constant mortality intensity by assuming that the time to death is distributed with an exponential distribution.

Now, we will verify the relationship of (55). Following Friedman (1957) and Hall (1978), the present value of future earnings is calculated by their discounted value at the risk-free interest rate. More specifically, with the stochastic life-cycle earnings $\epsilon(t)$ given by

$$\epsilon(t) = \begin{cases} \epsilon, & \text{if } 0 \leq t < \tau \wedge \tau^{\nu}, \\ k\epsilon, & \text{if } \tau \leq t < \tau^{\nu}, \\ 0, & \text{if } t \geq \tau^{\nu}, \end{cases}$$

the present value of future life-cycle earnings follows

$$
\begin{aligned}
&E\Big[\int_0^{\tau\wedge\tau^\nu} e^{-rt}\epsilon(t)dt + e^{-r(\tau\wedge\tau^\nu)}\int_{\tau\wedge\tau^\nu}^{\infty} e^{-r(t-\tau\wedge\tau^\nu)}\epsilon(t)dt\Big] \\
&= E\Big[\int_0^{\infty} \nu e^{-\nu s}\int_0^{\tau\wedge s} e^{-rt}\epsilon(t)dtds + \int_0^{\infty}\nu e^{-\nu s}e^{-r(\tau\wedge s)}\int_{\tau\wedge s}^{\infty} e^{-r(t-\tau\wedge s)}\epsilon(t)dtds\Big] \\
&= E\Big[\int_0^{\tau} \nu e^{-\nu s}\int_0^{s} e^{-rt}\epsilon dtds + \int_{\tau}^{\infty}\nu e^{-\nu s}\int_0^{\tau} e^{-rt}\epsilon dtds \\
&\qquad + \int_0^{\tau}\nu e^{-\nu s}e^{-rs}\int_s^{\infty} e^{-r(t-s)}0dtds + \int_{\tau}^{\infty}\nu e^{-\nu s}e^{-r\tau}\int_{\tau}^{s} e^{-r(t-\tau)}k\epsilon dtds\Big] \\
&= E\Big[\int_0^{\tau} e^{-rt}\epsilon\int_t^{\tau}\nu e^{-\nu s}dsdt + \int_0^{\tau} e^{-rt}\epsilon\int_{\tau}^{\infty}\nu e^{-\nu s}dsdt + \int_{\tau}^{\infty} e^{-rt}k\epsilon\int_t^{\infty}\nu e^{-\nu s}dsdt\Big] \\
&= E\Big[\int_0^{\tau} e^{-rt}\epsilon\int_t^{\infty}\nu e^{-\nu s}dsdt + \int_{\tau}^{\infty} e^{-rt}k\epsilon\int_t^{\infty}\nu e^{-\nu s}dsdt\Big] \\
&= E\Big[\int_0^{\tau} e^{-(r+\nu)t}\epsilon dt + \int_{\tau}^{\infty} e^{-(r+\nu)t}k\epsilon dt\Big] \\
&= \int_0^{\infty}\delta e^{-\delta s}\int_0^{s} e^{-(r+\nu)t}\epsilon dtdt + \int_0^{\infty}\delta e^{-\delta s}\int_s^{\infty} e^{-(r+\nu)t}k\epsilon dtds \\
&= \int_0^{\infty} e^{-(r+\nu)t}\epsilon\int_t^{\infty}\delta e^{-\delta s}dsdt + \int_0^{\infty} e^{-(r+\nu)t}k\epsilon\int_0^{t}\delta e^{-\delta s}dsdt \\
&= \int_0^{\infty} e^{-(r+\nu+\delta)t}\epsilon dt + \int_0^{\infty} e^{-(r+\nu)t}k\epsilon(1-e^{-\delta t})dt \\
&= \frac{1}{r+\nu+\delta}\epsilon + \frac{1}{r+\nu}k\epsilon - \frac{1}{r+\nu+\delta}k\epsilon \\
&= \frac{r+\nu+\delta k}{(r+\nu+\delta)(r+\nu)}\epsilon.
\end{aligned}
$$

Therefore, the lower bound L at time 0 in (55) derives from the multiplication of the present value stated above by exogenously given $\omega \in [0, 1)$ that represents the extent to which credit is tightened.

6 Further Numerical Results

Effects of Credit Tightening. Credit tightening affects the optimal consumption (Table 1) and investment (Table 2) strategies. Tightening of credit by decreasing ω makes individuals reduce their consumption amount; this response is especially significant for poor people, so their consumption smoothing is more difficult than for wealthy people. The effects of the LNIS worsens the situation for poor people. Given the significant downward jump in income in the aftermath of the LNIS, the poor people who are credit tightened would have difficulty to secure extra savings to finance their consumption needs. Hence, the consumption amount could fall further with the joint effects caused

$w \setminus \omega$	$\delta = 0$				$\delta = 0.07$				$\delta = 0.08$			
	0%	5%	10%	20%	0%	5%	10%	20%	0%	5%	10%	20%
1	1.0922	1.2090	1.2966	1.4336	0.7261	0.7200	0.7232	0.7281	0.7077	0.7033	0.7060	0.7098
10	1.8159	1.8646	1.9104	1.9948	1.1919	1.1809	1.1808	1.1810	1.1691	1.1578	1.1575	1.1586
20	2.4041	2.4402	2.4750	2.5409	1.6679	1.6538	1.6538	1.6539	1.6406	1.6285	1.6285	1.6285
30	2.9396	2.9698	2.9991	3.0552	2.1348	2.1183	2.1185	2.1185	2.1040	2.0915	2.0918	2.0910
40	3.4507	3.4772	3.5031	3.5528	2.5966	2.5787	2.5789	2.5789	2.5636	2.5507	2.5510	2.5501
50	3.9476	3.9715	3.9949	4.0400	3.0552	3.0365	3.0368	3.0368	3.0210	3.0075	3.0078	3.0069

$w \setminus \omega$	$\delta = 0.09$				$\delta = 0.10$			
	0%	5%	10%	20%	0%	5%	10%	20%
1	0.6862	0.6887	0.6909	0.6945	0.6748	0.6815	0.6784	0.6814
10	1.1396	1.1397	1.1397	1.1398	1.1223	1.1094	1.1233	1.1237
20	1.6071	1.6071	1.6072	1.6071	1.5894	1.5913	1.5890	1.5888
30	2.0680	2.0681	2.0681	2.0681	2.0498	2.0618	2.0489	2.0486
40	2.5260	2.5260	2.5260	2.5260	2.5069	2.5208	2.5059	2.5057
50	2.9820	2.9821	2.9821	2.9821	2.9620	2.9738	2.9613	2.9611

Table 1: **Optimal consumption amount for various credit tightening scenarios and intensity values of the large, negative shock.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate).

by the credit tightening and the income shock. Those effects also reduce the risky investment amount; this result is similar to the observation in the optimal consumption amount.

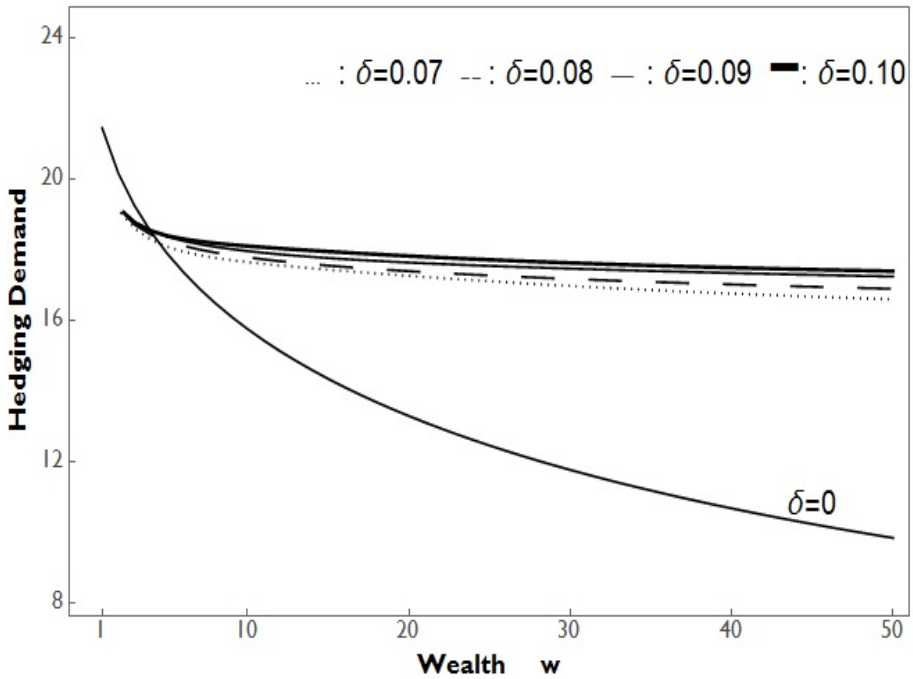

Figure 1: **Hedging demand.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate).

Hedging Demand. The amount of hedging demands differs between $\delta = 0$ and $\delta > 0$, and the differ-

$w \setminus \omega$	$\delta = 0$				$\delta = 0.07$				$\delta = 0.08$			
	0%	5%	10%	20%	0%	5%	10%	20%	0%	5%	10%	20%
1	4.0426	6.1272	7.6885	10.1335	5.2955	5.6019	5.8596	6.2650	5.3110	5.5845	5.8131	6.1575
10	14.2285	15.0976	15.9148	17.4194	12.3362	12.2797	12.8383	12.2981	12.2003	12.1282	12.1269	12.1598
20	21.6927	22.3378	22.9587	24.1342	17.7237	17.6876	17.6797	17.6806	17.5918	17.4760	17.4635	17.5022
30	28.2186	28.7577	29.2809	30.2818	23.0102	22.9245	22.9209	22.9207	22.8117	22.6988	22.6947	22.7050
40	34.3084	34.7818	35.2955	36.1304	28.2267	28.0941	28.0950	28.0951	27.9683	27.8593	27.8622	27.8523
50	40.1449	40.5720	40.9892	41.7945	33.3908	33.2257	33.2293	33.2298	33.0902	32.9808	32.9875	32.9675

$w \setminus \omega$	$\delta = 0.09$				$\delta = 0.10$			
	0%	5%	10%	20%	0%	5%	10%	20%
1	5.3298	5.5553	5.7509	6.0713	5.3526	5.7242	5.7158	5.9980
10	12.0225	12.0263	12.0312	12.0399	11.8807	11.6225	11.9189	11.9382
20	17.3480	17.3468	17.3471	17.3471	17.1623	16.6403	17.2001	17.2086
30	22.5196	22.5197	22.5197	22.5194	22.3573	22.2349	22.3573	22.3544
40	27.6439	27.6450	27.6452	27.6454	27.4934	27.6851	27.4720	27.4644
50	32.7435	32.7451	32.7453	32.7456	32.5915	32.9486	32.5630	32.5543

Table 2: **Optimal investment amount for various credit tightening scenarios and intensity values of the large, negative shock.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate).

ence increases as wealth w increases (Figure 1). This result implies that even sufficiently large wealth does not appropriately absorb the negative effects of the LNIS, and thereby amplifies the negative effects of background risk on risky investment compared to the positive effects of risk diversification.[1]

To obtain empirically plausible implications on investment, we carefully choose the parameter values for mortality rate and aggregate earnings by using the Survey of Consumer Finances (SCF) data. We adjust the values of mortality rate, ν, according to age of head given in the SCF. We assume that people die on average at the age of 80. For the age group of $35 - 44$, for instance, we vary ν from 0.0222 to 0.0278 at intervals of 0.0014. In this case, we have five values of ν. When matching up the stock investment with the SCF data, we take the median proportion of total wealth invested in stocks from age 35 through 44 (i.e., $\nu \in [0.0222, 0.0278]$) as the optimal investment proportion. When computing the optimal proportion of total wealth invested in stocks, we calibrate an initial endowment of financial wealth, w, and a constant stream of labor income per annum, $\epsilon = \xi I$, through the normalized cash-on-hand, i.e., the net worth normalized by income from the SCF.

The LNIS reduces the optimal portion of total wealth that should be invested in stocks (Table 3). The optimal investment strategy reported in the table differs from the wealth- and age- independent constant Merton (1969, 1971) investment rule. Our optimal investment strategy suggests that as people get older, their risky investment should be geared toward relatively safe assets; this advice is consistent with the rules of thumb proposed by financial advisers. The optimal risky portion itself

[1]There are two opposing motives on risky investment: a precautionary savings motive that reduces investment and a risk diversification motive that rises investment.

	Age				
Percentile of Net Worth	35-44	45-54	55-64	65-74	75-80
0-25	6.0	5.7	5.5	1.3	0.5
25-49.9	10.7	10.4	9.7	3.4	2.0
50-74.9	14.8	14.6	14.2	6.2	4.1
75-89.9	17.7	17.6	17.3	9.1	6.7
90-100	20.1	20.0	19.8	12.0	9.4
all	14.8	14.6	14.2	4.1	4.1

Table 3: **The optimal proportion of total wealth invested in stocks.**Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), and $k = 0.2$ (recovery rate). The values of mortality rate, ν, are adjusted following age of head given in the SCF, assuming that people die on average at the age of 80. An initial endowment of financial wealth, w, and a constant stream of labor income per annum, $\epsilon = \xi I$, are calibrated via the normalized cash-on-hand, i.e., the net worth normalized by income from the SCF for the period of 1995-2010. Note that the data period is chosen for including the 2007-2009 Great Recession in the U.S., when many people have experienced the unprecedented largest reductions in their consumption and unemployment.

is significantly $\leq$50%, as would be optimal in the Merton investment model. Our model generates empirically plausible values of 0 to 20% for optimal stock investment.

Interestingly, we show that people's risky investment ratio rises as their wealth increases, and this result is consistent with Wachter and Yogo (2010). The decision to invest in stocks is affected by two counteracting forces: a precautionary savings motive that decreases stock investment, and a diversification motive that increases stock investment. The presence of the LNIS itself increases undiversifiable background risk such as income risk, so precaution makes people conservative when taking on risk in the stock market. Accordingly, precautionary saving occurs. When saving occurs, the resources available for future investment are increased. Since the prices of risky investments are adjusted to increase their expected returns, the expected decline in labor income due to the LNIS can be partially offset. For risk diversification purposes, the share of total resources invested in the stock market would, thus, increase with an increase in wealth.

Utility Costs. We can measure utility costs of ignoring the LNIS as the wedge of value functions with and without the income shock. The costs can be thought of as the certainty equivalent wealth that is the greatest wealth the individual is willing to pay to reduce the probability or effect of the risk of catastrophic reduction in individual earnings.

Definition 6.1. *Let $\Delta(w)$ be the certainty equivalent wealth at initial wealth w, satisfying*

$$V(w - \Delta(w), I; \delta = 0) = V(w, I; \delta > 0),$$

where $V(w, I; \delta)$ is the value function given in (3) with the Poisson intensity δ.

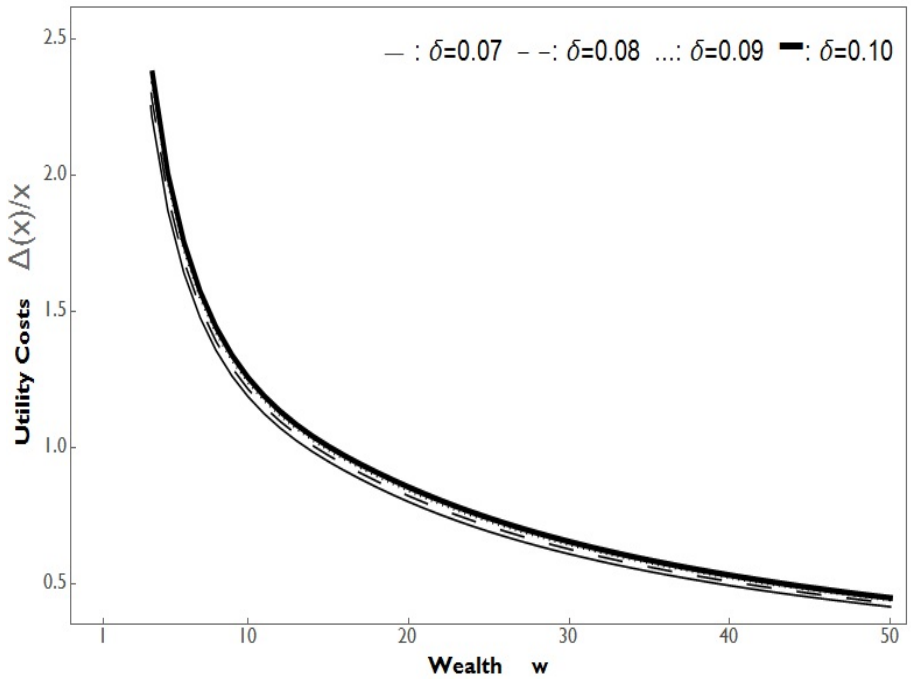

Figure 2: **Utility costs of ignoring a large, negative income shock**. Parameter Values: $\mu - r = 4\%$, $r = 2\%$, $\sigma = 20\%$, $\gamma = 2$, $\beta = 4\%$, $k = 20\%$, $L = 0$, $\epsilon = \xi I = 1$, and $\nu = 0.02$.

Ignoring the LNIS can result in substantial utility costs in the form of certainty equivalent wealth (Figure 2). Obviously, the maximum payment that the individual should be willing to accept to eliminate the possibility of the catastrophic income shock (or reduce its negative effects) increases with the probability that the shock will occur.

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